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Equilibrium pricing and market completion: a counterexample

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Abstract

In both arbitrage and utility pricing approaches, the fictitious completion appears as a powerful tool that permits to extend complete markets results to an incomplete markets framework. Does this technique permit to characterize the equilibrium pricing interval? This note provides a negative answer.

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1 Introduction

Asset pricing theory and its link to arbitrage have been formalized by Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983). The main result in these models is that the price process of traded securities is arbitrage free if and only if there exists some equivalent probability measure that transforms it into a martingale, when normalized by the numeraire. When markets are dynamically complete, there is only one such a martingale-probability measure and any contingent claim is priced by taking the expected value of its (normalized) payoff with respect to this measure. It is said priced by arbitrage. When markets are incomplete, arbitrage bounds can be computed taking the expected value of the (normalized) payoff with respect to all the measures that characterize the absence of arbitrage. The obtained arbitrage bounds correspond to the minimum amount it costs to hedge the claim and the maximum amount that can be borrowed against it using dynamic strategies. Each one of the measures that characterize the absence of arbitrage can be seen as associated to a specific *fictitious completion*, that is to say a complete financial market that consists of the initial market completed by additional nonredundant assets. Each price within the arbitrage bounds corresponds then to the arbitrage price in a specific *fictitious completion* and the set of such prices is called arbitrage pricing interval. This line of research has been initiated by El Karoui and Quenez (1995).

Utility pricing is another important approach. The price of a given risky claim, following this approach,corresponds to the amount for which an agent whose preferences and initial endowment are specified is indifferent, at his optimal consumption plan, between a positive or a negative additional marginal quantity of the claim for that unitary amount. When markets are complete, Pliska (1986), Cox and Huang (1989, 1991), Karatzas *et al.* (1990) adapted the martingale ideas presented above to problems of utility maximization. When markets are incomplete, Karatzas *et al.* (1991) builds upon these last references in order to compute a utility price for every *fictitious completion*, and show that the utility price in the initial incomplete market corresponds to the utility price in the completion which makes the maximum expected utility as small as possible. For a given specification of agent's preferences (utility function and initial endowment) and for a given risky claim, this approach leads to a unique price. However, this price strongly depends on the chosen specification. In order to obtain preferences and endowments-free bounds, it is possible, for a given risky claim, to introduce the concept of utility pricing interval (whose bounds are called utility bounds). This interval is the set of utility prices that are obtained for some utility function in the class of von Neumann-Morgenstern (vNM) increasing and concave utility functions and for some initial endowment. Jouini and Kallal (1999) show that the utility bounds (utility pricing interval) coincide with the arbitrage bounds (arbitrage pricing interval). In the next we will call them arbitrage/utility bounds (arbitrage/utility pricing interval, AUPI).

In both arbitrage and utility pricing approaches, the *fictitious completion* appears as a very powerful tool that permits to generalize results established in a complete markets setting to an incomplete markets framework. However, the arbitrage/utility pricing interval is often too large to be of great interest (Cvitanic *et al.*, 1999).

It is then necessary to introduce additional restrictions in order to derive tighter bounds. The equilibrium pricing approach has been introduced first by Bizid *et al.* (1998) and makes an explicit use of the market clearing conditions. In a general setting and for a given risky claim, equilibrium bounds delimit the set of all possible equilibrium prices for the asset under consideration for all possible distributions of agents preferences (within the class of vNM increasing and strictly concave utility functions) and for all possible ditributions of initial endowment across the agents. Bizid and Jouini (2005) show that this interval is smaller than the arbitrage/utility pricing one even though they do not fully characterize it.

At this stage, a natural question arises: does the completion technique permit to characterize the equilibrium pricing interval (*EPI*) as the set of equilibrium prices associated to all possible completions (that we will call *fictitious completion* equilibrium pricing interval, *FCEPI*)? In other words, does the set of prices that can be reached at the equilibrium for at least one distribution of preferences/endowments and for at least one completion coincide with the set of prices that can be reached at the equilibrium for at least one distribution of preferences/endowments?

The question above is quite important because the *FCEPI* corresponds to the pricing interval considered in Bizid *et al.* (1998), Jouini (2003) and Jouini and Napp (2003) and, as shown therein, is relatively easy to determine and has many interesting properties in terms of size and robustness.

Unfortunately, the answer is negative. The present note properly defines these different concepts and provides a simple example where the set of equilibrium prices associated to all possible vNM utility functions is strictly larger than the set of equilibrium prices associated to all possible *fictitious completion*s.

The paper is organized as follows. Section 2 presents the framework and the main results. Section 3 presents the specific example. Section 4 concludes. Proofs are in the Appendix.

2 The model

We consider a model with 2 dates, a finite set of states of the world Ω, endowed with a probability P whose expectation operator is denoted by $E[.]$. Two random variables x and y are anticomonotonic (denoted by $x \downdownarrows y$) if

$$\text{for all } (\omega, \omega') \in \Omega^2,\ x(\omega) > x(\omega') \text{ if and only if } y(\omega) > y(\omega').$$

Consumption takes place at date $t = 1$. There are K productive assets. The k^{th} asset has a price p^k and pays a random dividend $d^k(\omega)$. There are also M derivative assets and the m^{th} asset has a price q^m and delivers a random amount f^m. All prices are determined at date 0 but transfers occur at date 1.

There are N agents with utility functions $U^n(.) = E[u^n(C)]$ where $u^n : \mathbb{R}_+ \setminus \{0\} \rightarrow \mathbb{R}$. At $t = 0$, agent n has $\theta_0^n = \left(\theta_0^{k,n}\right)_{k=1,...,K}$ shares of the productive assets, $\alpha_0^n = (\alpha_0^{m,n})_{m=1,...,M}$ shares of the derivative assets and a random endowment w^n. We have $\sum_{n=1}^N \theta_0^n = (1, ..., 1)$, $\sum_{n=1}^N \alpha_0^n = 0$.

Assumption U. u^n is continuously differentiable, increasing, strictly concave and satisfies $u^n(x) \underset{x \rightarrow 0^+}{\rightarrow} -\infty$.

We consider a new derivative asset whose payoffs are described by f_0 (in short, asset f_0). At equilibrium, which prices q_0 for f_0 are compatible. with the already observed prices $\left(\left(p^k\right),\left(q^m\right)\right)$?

Let us first introduce the following definitions.

A strategy S is a vector (C,θ,α) where $C\geq 0$ is a random consumption and where θ (α) are quantities of productive (derivatives) assets.

The budget constraint of the n^{th} agent is then given by:

$$\begin{aligned}\theta^n\cdot p+\alpha^n\cdot q &= \theta_0^n\cdot p+\alpha_0^n\cdot q, \\ C &= w^n+\theta^n\cdot d+\alpha^n\cdot f.\end{aligned} \tag{1}$$

where $\cdot$ denotes the inner product.

For a given agent n with an initial endowment $(w^n,\theta_0^n,\alpha_0^n)$, the set $\mathcal{A}^n$ is the set of strategies S satisfying the budget constraint (1).

As usual, an equilibrium and a state-price deflator (SPD) are defined as follows:

Definition 1. An equilibrium in the economy $\mathcal{E}=\left(\left(d,f\right),\left(u^n,w^n,\theta_0^n,\alpha_0^n\right)_{n=1,\ldots,N}\right)$ is defined by strategies $(C^{n,*},\theta^{n,*},\alpha^{n,*})$ and prices (p,q) such that, for $n=1,...,N$, (C^n,θ^n,α^n) maximizes U^n on $\mathcal{A}^n$, $\sum_{n=1}^N\theta^n=(1,...,1)$, $\sum_{n=1}^N\alpha^n=0$ and $\sum_{n=1}^N C^n=w+D$ where $w=\sum_{n=1}^N w^n$ and $D=\sum_{k=1}^K d^k$.

Definition 2. A SPD is a random variable $\varsigma>0$ such that

$$p=E\left[\varsigma d\right] \text{ and } q=E\left[\varsigma f\right].$$

We assume that the aggregate endowment w is known and that agents observe assets' prices. However, they do not know the number of agents nor their individual characteristics (utility functions, endowments).

For a given derivative asset, we want to characterize the prices that might emerge at equilibrium for some configuration of agents' preferences and some distribution of the initial endowment across the agents.

Definition 3. $EPI(f^0)$, is the set of prices q^0 for which there exists an integer N, utility functions $(u^n)_{n=1,\ldots,N}$ satisfying (U), initial endowments $(w^n,\theta_0^n,\alpha_0^n)_{n=1,\ldots,N}$ and strategies $\left(C^{n,*},\theta^{n,*},(\alpha^{0,n,*},\alpha^{n,*})\right)$, such that $\left(C^{n,*},\theta^{n,*},(\alpha^{0,n,*},\alpha^{n,*})\right)$ and $(p,(q^0,q))$ define an equilibrium of the economy $\widetilde{\mathcal{E}}=\left(d,\left(f^0,f\right),\left(u^n,w^n,\theta_0^n,\alpha_0^n\right)_{n=1,\ldots,N}\right)$.

The following proposition provides a simpler characterization of the *EPI*.

Proposition 1 *$EPI(f_0)$ is the set of prices q^0 for which there exists N, positive random vectors $(C^n)_{n=1,\ldots,N}$ and SPDs $(\varsigma^n)_{n=1,\ldots,N}$, such that $\sum_{n=1}^N C^n=w+D$, $C^n\downdownarrows\varsigma^n$, and $q^0=E\left[\varsigma^n f^0\right]$, for $n=1,...,N$, i.e.*

$$EPI(f_0)=\bigcup_{N>0}\ \bigcup_{C^1+\ldots+C^n=w+D}\ \bigcap_{n=1,\ldots,N}\left\{E\left[\varsigma f_0\right]:\varsigma^n \text{ is a SPD and } \varsigma\downdownarrows C^n\right\}.$$

When markets are complete and all information is available, there is only one SPD ς, and $EPI(f_0) = \{E[\varsigma f^0]\}$.

In complete markets, if only some prices are observable, there is only one SPD but it can not be determined due to information incompleteness. However, the comonotonicity condition partially or fully compensates the lack of information on assets prices as shown by Bizid *et al.* (1998), Jouini and Napp (2002) or Chazal and Jouini (2008) (see also Perrakis and Ryan, 1984, Ritchken, 1985, Perrakis, 1986). In an incomplete markets setting, Bizid and Jouini (2005) shows that the *EPI* is strictly smaller than the *AUPI*.

A *fictitious completion* is described by J additional derivatives $(f^{M+j})_{j=1,...,J}$ with prices $(q^{M+j})_{j=1,...,J}$ such that the market that consists in the K productive assets and the $M+J$ derivatives, is complete. The *FCEPI* is defined as follows.

Definition 4. $FCEPI(f^0)$, is the set of prices q^0 such that there exists a *fictitious completion* of the initial market such that, in this completed market, q^0 is in $EPI(f^0)$.

Proposition 2 $FCEPI(f_0) = [\inf_{\varsigma\in\Xi} E[\varsigma f_0], \sup E_{\varsigma\in\Xi}[\varsigma f_0]]$ *where* Ξ *is the set of all SPDs* ς *such that* $\varsigma \downdownarrows w + D$.

In a diffusion setting, Jouini and Napp (2003) show that the *FCEPI* is reduced to a unique price that corresponds to Föllmer and Schweizer (1991) price when equity prices are increasing functions of dividends[1]. Recall that Föllmer and Schweizer price corresponds to the price when there is a 0 premium for risks that are orthogonal to the already existing market.

If the *EPI* coincides with the *FCEPI*, this would provide a simpler characterization of the EPI and would also mean that the EPI is robust and very tight.

In general, do we have $EPI(f_0) = FCEPI(f_0)$ or, equivalently,

$$\inf_{\varsigma \text{ SPD and } \varsigma\downdownarrows w+D} E[\varsigma f_0] = \inf_{N>0} \inf_{C^1+...+C^n=w+D} \sup_{n=1,...,N} \inf_{\varsigma \text{ SPD and } \varsigma\downdownarrows C^n} E[\varsigma f_0]$$

for all f_0?

The next section provides a negative question to this question.

3 A Specific example

We take $|\Omega| = 4$, $K = 1$, $M = 0$, $d^1 = (1, 1, 1, 1)$, $p^1 = 1$, $w = (2.997, 2.52, 2.501, 2)$ and $f^0 = (1, 0, 0, \frac{1}{10})$.

For $N = 2$, $w^1 = (1.499, 1.5, 1.001, 1)$, $w^2 = (1.498, 1.02, 1.5, 1)$, $\varsigma^1 = (1.1375, 0.5025, 1.14, 1.22)$ and $\varsigma^2 = (1.1373, 1.1415, 0.4992, 1.222)$, we check that $w^n \downdownarrows \varsigma^n$ for $n = 1, 2$, $w^1 + w^2 = w$, $E[\varsigma^1 d^1] = E[\varsigma^2 d^1] = p^1$ and $E[\varsigma^1 f^0] = E[\varsigma^2 f^0] = 0.314875$.

By Proposition 1, we have $\bar{q}^0 = 0.314875 \in EPI(f^0)$.

By Proposition 2, $FCEPI(f^0) = [\inf E[\varsigma D], \sup E[\varsigma D]]$ for ς such that $E[\varsigma^1 d^1] = p^1$ and $\varsigma \downdownarrows w$. For such a ς, it is easy to check that we necessarily have $\frac{1}{30} \leq E[\varsigma D] \leq 0.275$

[1]Jouini and Napp (2003) provides theoretical justifications for such an assumption.

and any value within these bounds can be reached by such a ς. Hence, $\bar{q}^0 \notin FCEPI(f^0) = [1/30, 0.275]$.

Note that $AUPI\,(f^0)$ is given by $[\inf E\,[\zeta f^0]\,, \sup E\,[\zeta f^0]]$ for ζ such that $E\,[\varsigma] = 1$ hence $AUPI\,(f^0) = [0,1]\,.$

Any price $q^0 \in EPI(f^0)$ is the equilibrium price in a given economy. Since we have $w_1 > w_2$, at least one agent satisfies $C_1^{n,*} > C_2^{n,*}$ which gives $\zeta_1^n < \zeta_2^n$ for the associated SPD hence $q^0 \leq 0.5$.

We have then $\left[\frac{1}{30}, \frac{11}{40}\right] = FCEPI(f^0) \subsetneq EPI(f^0) \subset [0, 0.5] \subsetneq AUPI\,(f^0) = [0,1]\,.$

4 Conclusion

Bizid and Jouini (2005) shows that the pricing interval obtained through partial equilibrium conditions is strictly smaller than the *AUPI* and strictly larger than the *FCEPI*. However, it was unclear if the full exploitation of the equilibrium conditions would permit to reach a smaller pricing interval (*EPI*) and whether this interval would be equal to the *FCEPI*. This question is interesting because the *FCEPI* might be quite tight, has good convergence properties and is robust to small perturbations on the characteristics of the economy. We have shown that the *EPI* might be strictly larger than the *FCEPI* : the fictitious completion technique does not permit to scan all possible equilibrium prices while it permits to scan all arbitrage/utility prices. Then, work remains to be done in order to analyze to which extent the bounds of Bizid and Jouini (2005) can be reduced further.

The main conclusion of this note is that the *fictitious completion* technique does not permit to scan all possible equilibrium prices while it permits to scan all arbitrage/utility prices.

5 Appendix

Proof of Proposition 1. If $q^0 \in EPI(f^0)$ then there exists N, $(u^n)_{n=1,...,N}$, $(w^n, \theta_0^n, \alpha_0^n)_{n=1,...,N}$ and $(C^{n,*}, \theta^{n,*}, (\alpha^{0,n,*}, \alpha^{n,*}))$ such that $((C^{n,*}, \theta^{n,*}, (\alpha^{0,n,*}, \alpha^{n,*}))\,; (p, (q^0, q)))$ is an equilibrium of $\widetilde{\mathcal{E}} = (d, (f^0, f)\,, (u^n, w^n, \theta_0^n, \alpha_0^n)_{n=1,...,N})$. The first order conditions for Agent 1 utility maximization give

$$\left(u^1\right)'\left(C^{1,*}\right) d^k = \lambda p^k \text{ and } \left(u^1\right)'\left(C^{1,*}\right) f^m = \lambda q^m$$

for $k = 1, ..., K$, $m = 0, ..., M$ and for some Lagrange multiplier λ. It suffices to take $\varsigma = \frac{1}{\lambda}\left(u^1\right)'\left(C^{1,*}\right).$

Conversely, let us assume that there exists $(C^n)_{n=1,...,N}$ and SPDs $(\varsigma^n)_{n=1,...,N}$ such that $\sum_{n=1}^N C^n = w + D$, $C^n \downdownarrows \varsigma^n$, and $q^0 = E\,[\varsigma^n f^0]$ for $n = 1, ..., N$. For a given n, let us number the elements of Ω such that $C^n(\omega_i)$ is nondecreasing and let us define the function v^n such that $v^n(c) = \varsigma^n(\omega_1)\frac{C^n(\omega_1)}{c}$ on $(0, C^n(\omega_1)]$, v^n affine on $[C^n(\omega_i), C^n(\omega_{i+1})]$ for $1 \leq i \leq |\Omega| - 1$, $v^n(c) = \varsigma^n\left(\omega_{|\Omega|}\right)\frac{C^n\left(\omega_{|\Omega|}\right)}{c}$ on $\left[C^n\left(\omega_{|\Omega|}\right), \infty\right)$, and v continuous. It is immediate that v is positive, decreasing, that $u^n(c) = \int_1^c v^n(x)dx$ satisfies (U) and that $(u^n)'(C^n) = \varsigma^n$. Let us take $w^n = C^{n,*} - \frac{1}{N}D$, $\theta^{0,n} = \frac{1}{N}\mathbf{1}_K$ and $\alpha^{0,n} = 0$ for $n = 1, ..., N$. It is immediate that $\left(C^n, \theta^{0,n}, (\alpha^{0,n}, \alpha^n)\right)$ satisfies the first order necessary and sufficient

conditions associated to u_n, $n = 1, ..., N$, in the market $((d,p), ((q^0, q), f^0, f))$. Since we also have, by construction, $\sum_{n=1}^{N} C^{n,*} = w + D$, $\sum_{n=1}^{N} \theta^{n,*} = (1, ..., 1)$, $\sum_{n=1}^{N} \alpha^{n,*} = 0$ then $((C^{n,*}, \theta^{n,*}, (\alpha^{0,n,*}, \alpha^{n,*})); (p, (q^0, q)))$ is an equilibrium of $\widetilde{\mathcal{E}} = (d, (f^0, f), (u^n, w^n, \theta_0^n, \alpha_0^n)_{n=1,...,N})$.
■

Proof of Proposition 2. If $q^0 \in FCEPI(f_0)$, there exists a completion for which q^0 is in the associated EPI. By proposition 1, there exists $(C^n)_{n=1,...,N}$ and SPDs $(\varsigma^n)_{n=1,...,N}$, s.t. $C^n \downdownarrows \varsigma^n$ and $\sum_{n=1}^{N} C^n = w + D$. Since it is a completion, there is only one SPD. We have then $\varsigma^1 = ... = \varsigma^n \downdownarrows \sum_{i=1}^{n} C^i = w + D$. Conversely, if $q^0 = E[\varsigma f^0]$ for a SPD $\varsigma \downdownarrows w + D$, let us complete our market with enough additional derivatives $(f^{M+j})_{j=1,...,J}$ with prices $q^{M+j} = E\left[\varsigma f^{M+j}\right]$, $j = 1, ..., J$. By Proposition 1, $q^0 \in EPI(f^0)$ in this completion.
■

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