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Notes

On multivariate prudence ☆

Elyès Jouini [a], Clotilde Napp [b,c], Diego Nocetti [d,*]

[a] *PSL, Université Paris-Dauphine, CEREMADE, France*
[b] *CNRS, UMR7088, France*
[c] *PSL, Université Paris-Dauphine, DRM, France*
[d] *School of Business, Clarkson University, United States*

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Abstract

In this note we extend the theory of precautionary saving to the case of multivariate risk. We introduce a notion of *multivariate prudence*, related to a precautionary premium, and we propose a matrix-measure to capture the strength of the precautionary saving motive. We discuss the usefulness of this measure, in particular for comparing precautionary behavior among individuals. We also characterize the notion of *multivariate downside risk aversion* as a preference for disaggregating harms across outcomes of multivariate lotteries. We show the link between this notion and the notion of multivariate prudence, we propose a matrix-measure of its intensity, and we illustrate the usefulness of our results in a problem of social discounting.

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1. Introduction

It has been known since Leland [18] and Sandmo [23] that a positive third derivative of a von Neumann–Morgenstern utility function is equivalent to a precautionary saving motive;

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* Corresponding author at: Clarkson University, PO Box 5790, Potsdam, NY 13699-5790, United States.
E-mail addresses: jouini@ceremade.dauphine.fr (E. Jouini), clotilde.napp@dauphine.fr (C. Napp), dnocetti@clarkson.edu (D. Nocetti).

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that is, an increase in saving in response to the presence of uncertainty surrounding future income. Kimball [16] gave the name *prudence* to the sensitivity of optimal saving to risk and proposed the index of prudence, $\frac{-v'''(x)}{v''(x)}$, as a measure of this sensitivity.

A positive third derivative of the utility function has also been shown to be equivalent to a preference for disaggregating harms across lottery outcomes known as *downside risk aversion* [9,20]; that is, a preference for lotteries in which a mean-zero risk is present when the level of wealth is high rather than when the level of wealth is low. As far as the intensity is concerned, however, it has been common to use the index $\frac{v'''(x)}{v'(x)}$ to measure the strength of downside risk aversion [4,6,13,21].

While these two measures of prudence and of downside risk aversion are tremendously useful in many contexts, one of their drawbacks is that they can only be applied to situations in which utility is a function of one variable (e.g. wealth). They cannot be applied, for example, to analyze the intensity of precautionary saving in models with multidimensional risks (as in [3,7,19]), or to analyze the intensity of preferences over multidimensional lotteries (as in [10]). Fortunately, it will be shown in this note that Karni's [14] generalization to a multivariate setting of the theory of risk aversion developed by Pratt [22] and Arrow [1] can be readily extended to evaluate the intensity of higher-order multivariate risk attitudes.

In the first part of this note we study the sensitivity of optimal saving to a multivariate risk in a 2-period model where monetary investments at the first period may have multidimensional consequences at the second period. We extend Kimball's [16] analysis of prudence to a multidimensional setting. We begin by introducing a notion of *multivariate prudence*, which is related to precautionary saving behavior in a multidimensional setting. As done by Kimball [16] in the univariate setting, we highlight the isomorphism between the theory of precautionary saving and the theory of risk aversion. Then, we propose to measure the intensity of multivariate prudence using Karni's [14] methods on multivariate risk aversion. We show, in particular, that applying Karni's matrix-measure of multivariate risk aversion to the negative of the marginal utility of saving permits to make comparisons of the intensity of precautionary saving among individuals, just as Kimball's [16] measure $\frac{-v'''(x)}{v''(x)}$ does in the univariate setting. Following the analysis of Drèze and Modigliani [8] and Kimball [16] in a univariate setting, we also evaluate the strength of the precautionary saving motive in a multivariate setting relative to the strength of multivariate risk aversion.

We emphasize that our analysis extends and complements not only the work of Kimball [16] but also more recent work [3,7,19] that establish conditions for precautionary saving to occur in the context of bivariate utility functions.

The second part of this note is devoted to study a more primitive attitude towards risk that we label *multivariate downside risk aversion* (MDRA). In analogy to its univariate counterpart, we say that an individual displays MDRA if he prefers to "disaggregate harms" and to locate a multidimensional risk to states of nature in which the level of the attributes is high rather than to states of nature in which the level of the attributes is low. In an expected utility framework, we show the equivalence between the notion of multivariate downside risk aversion and the notion of multivariate prudence. As Crainich and Eeckhoudt [4] do in a univariate setting, we propose a (matrix) measure of MDRA that is related to the amount of money necessary to compensate for the misallocation of the risk and we analyze its usefulness. Finally, we apply our results to a problem of social discounting analogous to Gollier [11].

The different parts of the note follow the structure delineated in the preceding discussion.

2. Precautionary saving with multivariate risk

2.1. The setting

We consider a 2-period model. The individual derives utility from $(n+1)$ attributes, and is endowed with $(n+1)$-dimensional, increasing (with respect to each attribute) and concave first and second period utility functions u and v. The first attribute is the income. Many interpretations are possible for the other variables, including a vector of market prices, non-traded commodities (e.g. health status, environmental quality), or social attributes (e.g. health of other members of the family, income differences with the "neighbors"). We let $x_0 = (x_{00}, \dots, x_{0n})$ and $x_1 = (x_{10}, \dots, x_{1n})$ denote the initial endowments of the individual in the $(n+1)$ attributes respectively at the first and second period.

In the first period, the individual saves an amount s of income (hence consumes an amount $x_{00} - s$ of income). We assume that this saving may impact not only the income at the second period but also the other attributes. We let $\rho = (\rho_0, \dots, \rho_n)$ denote the $(n+1)$-dimensional rate of return, which means that saving an amount s of income in the first period enables the individual to get $\rho_0 s$ of income as well as $\rho_i s$ of the ith attribute, for $i = 1, \dots, n$, at the second period. For example, a consumer may invest current resources in improving his or her future health (as Denuit, Eeckhoudt, and Menegatti [7] recently argued), and future health may in turn have positive externalities on the future income. At the aggregate level, investing and therefore consuming less also corresponds to producing less and might have a positive impact on the environment. Similarly, investing in environmental projects leads to financial returns (as for any project) but also to environmental improvement.

Alternatively, this setting permits to model different types of saving in exogenous proportions, e.g. a proportion of saving is devoted to "market" saving, another proportion to preventive health care, and so on, in which case we can think of ρ_i as the exogenous proportional rate of return on total saving s. In socially responsible investment, a proportion of the invested amount might be devoted to social projects (e.g. poverty fighting), leading to an improvement of the social responsibility feeling, and the rest of the amount might be invested in financial markets, leading to classical financial returns. We emphasize, however, that our setting also embeds the more traditional bivariate models of precautionary saving [3,7,10,19], in which current monetary investments only have monetary consequences.[1]

We assume the following regularity and Inada conditions on the utility functions.

Condition C1.

1. *Regularity*: u and v are C^3.
2. *Inada*[2] *on* u: $\lim_{x_{00}\to-\infty} u_0(x_{00}, \dots, x_{0n}) = \infty$ and $\lim_{x_{00}\to\infty} u_0(x_{00}, \dots, x_{0n}) = 0$, where $(x_{01}, \dots, x_{0n})$ is kept fixed.

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[1] In the bivariate models of Eeckhoudt et al. [10], Courbage and Rey [3] and Menegatti [19] it is assumed that $\rho_0 = 1$ and $\rho_1 = 0$. Denuit et al. [7] also analyze the case with $\rho_1 = 1$ and $\rho_0 = 0$.

[2] We adopt the following standard notations. For a function f of $(n+1)$ variables, $f_i \equiv \frac{\partial f}{\partial x_i}$ and $f_{ij} \equiv \frac{\partial^2 f}{\partial x_i \partial x_j}$. For a vector x in $\mathbb{R}^{n+1}$, we let x_{-i} denote the vector x deprived of its ith coordinate.

3. *Uniform Inada on* v: for each i such that $\rho_i \neq 0$, $\lim_{x_{1i} \to -\infty} v_i(x_{10}, \dots, x_{1n}) = \infty$, where the convergence is uniform with respect to $x_{1,-i}$ and $\lim_{x_{1i} \to \infty} v_i(x_{10}, \dots, x_{1n}) = 0$, where the convergence is uniform with respect to $x_{1,-i}$ on $[k, \infty)$ for some[3] k.

We introduce the function $w \equiv \nabla v \cdot \rho$. The function w represents the multivariate second period marginal utility of saving and will reveal to be of particular importance for our analysis. For simplicity of notation, we introduce the vector $\delta \equiv (1, 0, \dots, 0)$.

Under certainty, the consumption/saving problem is

$$\max_{s \in \mathbb{R}} h(s) \tag{1}$$

with

$$h(s) = u(x_{00} - s, x_{01}, \dots, x_{0n}) + v(x_{10} + \rho_0 s, \dots, x_{1n} + \rho_n s).$$

We have

$$h'(s) = -u_0(x_0 - s\delta) + \sum_{i=0}^{n} \rho_i v_i(x_1 + s\rho) = -u_0(x_0 - s\delta) + w(x_1 + s\rho).$$

Under Condition C1, it is easy to verify that there exists a unique s^* in $\mathbb{R}$ such that $h'(s^*) = 0$ and that s^* solves the consumption/saving problem (1). It is also easy to verify that the optimal solution s^* is increasing in x_{00}; that is to say, the optimal saving level is increasing with first period income, which is a natural property. We further impose the natural analogous condition that the optimal solution s^* is decreasing in x_{10}, that is to say, the optimal saving level is decreasing with second period income. It is easy to verify that this condition is equivalent to

Condition C2. The second period marginal utility of saving is decreasing in the first variable, i.e., $w_0 < 0$ on $\mathbb{R}^{n+1}$.

Consider now the case with multivariate risk. There is noise $\tilde{e} = (\tilde{e}_0, \dots, \tilde{e}_n)$ affecting the vector x_1 of second period consumption, where $E[\tilde{e}] = 0$. We denote by $\tilde{x}_1 \equiv x_1 + \tilde{e}$ the vector of second period noisy consumption and by $V^e \equiv [\sigma_{ij}]$ with $\sigma_{ij} = cov(\tilde{e}_i, \tilde{e}_j)$, the $(n+1) \times (n+1)$ variance–covariance matrix of $\tilde{e}$. The consumption/saving problem becomes

$$\max_{s \in \mathbb{R}} H(s) \tag{2}$$

with

$$H(s) = u(x_{00} - s, x_{01}, \dots, x_{0n}) + E\big[v(\tilde{x}_{10} + \rho_0 s, \dots, \tilde{x}_{1n} + \rho_n s)\big].$$

The solution is denoted by $\hat{s}$ and, under Conditions C1 and C2, it is characterized by $H'(\hat{s}) = 0$ or equivalently $u_0(x_0 - \hat{s}\delta) = E[w(\tilde{x}_1 + \hat{s}\rho)]$. The analysis of precautionary saving in a multivariate setting consists in comparing s^* and $\hat{s}$.

[3] When v is interpreted as an indirect utility function where x_{10} is a wealth variable and where $(x_{11}, \dots, x_{1n})$ are price variables, it is natural to assume that $\rho_i = 0$, $i = 1, \dots, n$, and the Inada condition is then only on v_0.

2.2. *Precautionary saving, multivariate prudence and a matrix-measure of multivariate prudence*

In the spirit of Kimball [16], we propose a definition of prudence in the multivariate setting which is directly related to precautionary saving behavior.

Definition 1. An individual with utility functions (u, v) satisfying Conditions C1 and C2 is multivariate prudent – in the direction of ρ – if any additional multivariate risk $\tilde{e}$ generates precautionary saving for all initial endowments, i.e. $\hat{s} \geqslant s^*$ for all (x_0, x_1).

In Kimball [16], the precautionary premium ψ is defined by $E[v'(\theta_0 + \tilde{\theta})] = v'(\theta_0 - \psi)$, where v denotes the univariate second period utility function. The precautionary premium then corresponds to the certain reduction of second period income that has the same upward effect on the optimal level of first period saving as the introduction of the additional risk. In the same spirit, we define the precautionary premium $\psi(x, \tilde{e}, w)$ in the multivariate setting as follows

$$E\big[w(\tilde{x})\big] = w\big(x - \psi(x, \tilde{e}, w)\delta\big) \equiv \nabla v \cdot \rho\big(x_0 - \psi(x, \tilde{e}, w), x_1, \ldots, x_n\big). \tag{3}$$

We shall confine the discussion to the case where the expectation on the left-hand side of (3) is finite. The existence and uniqueness of ψ then follows from the fact that w is continuous and decreasing. The individual is multivariate prudent in the sense of Definition 1 if and only if the premium ψ is nonnegative.

Notice that our precautionary premium, as defined by Definition 1, corresponds to the multivariate risk premium defined[4] by Karni [14], replacing the (indirect) utility function v by the negative of the multivariate marginal utility of saving $-w \equiv -\nabla v \cdot \rho$. This means that the analogy between the theory of precautionary saving and the theory of risk aversion, highlighted by Kimball [16] in the univariate setting, extends to the multivariate setting in the following way: in the univariate (multivariate) setting, the negative of marginal utility, $-v'$ (of marginal utility of saving, $-w \equiv -\nabla v \cdot \rho$), plays substantially the same role for precautionary saving that the univariate (multivariate) utility function itself v plays for risk aversion. This analogy being obtained, we can apply Karni's [14] methods and results about multivariate risk aversion in order to analyze precautionary saving in a multivariate setting. We first propose as a measure of (local) multivariate prudence Karni's [14] matrix-measure of (local) risk aversion, $A^v \equiv [-\frac{v_{ij}}{v_0}]$, replacing v by $-w$.

Definition 2. The matrix-measure of multivariate prudence is defined by $P^w \equiv [-\frac{w_{ij}}{w_0}]$.

By Taylor expansions of the functions on both sides of (3), we obtain the following local property.

Proposition 1. *A local approximation of the precautionary premium can be expressed as* $\psi(x, \tilde{e}, w) \approx -\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{w_{ij}(x)}{w_0(x)}\sigma_{ij}$ *or equivalently* $\psi(x, \tilde{e}, w) \approx \frac{1}{2}\,tr[V^e P^w(x)]$.

[4] For an individual with a utility function v and initial endowment x, Karni [14] defines the risk premium $\Pi(x, \tilde{e}, v)$ as the certain reduction of income (first attribute) that has the same effect on utility as the introduction of a multivariate risk $\tilde{e}$, i.e. such that $E[v(\tilde{x})] = v(x_0 - \Pi(x, \tilde{e}, v), \ldots, x_n)$.

The diagonal elements of P^w, $-\frac{w_{jj}}{w_0}$, capture the precautionary premium per unit of variance σ_{jj}, when each risk is taken in isolation; the first diagonal element of P^w, $-\frac{w_{00}}{w_0}$, can be interpreted as a measure of "own prudence" while each of the other diagonal elements of P^w, $-\frac{w_{jj}}{w_0}$, can be interpreted as measures of "cross-prudence" [10]. The off-diagonal elements, $-\frac{w_{ij}}{w_0}$, can be interpreted as the excess precautionary premium per unit of covariance; they capture the distaste for positive dependence between two risks, which the individual compensates with extra saving.

Proposition 2. *The following conditions are equivalent*:

1. *The marginal utility of saving function w is convex.*
2. *The precautionary premium is nonnegative, i.e. $\psi(x, \tilde{e}, w) \geqslant 0$, for all $(x, \tilde{e})$.*
3. *The matrix-measure of multivariate prudence $P^w(x)$ is positive semidefinite for all x.*

Proof. The proof follows the lines of the proof of Theorem 1 in Karni [14] replacing the (indirect) utility function v by the negative of the marginal utility of saving, $-w$, and using the fact that w is decreasing in its first variable (Condition C2). □

The property that the multivariate precautionary premium is nonnegative if and only if our matrix-measure of multivariate prudence is positive semidefinite is important.[5] However, possibly more important is to establish if our matrix-measure of multivariate prudence can be used for comparing the intensity of precautionary savings among individuals.

2.3. Comparative multivariate prudence

Kimball [16] established that in the presence of a unidimensional risk, Agent A will require a larger precautionary premium than Agent B if and only if his index of prudence is larger, i.e., $-\frac{v_A'''}{v_A''} \geqslant -\frac{v_B'''}{v_B''}$.

In our multivariate setting, Agent A, with utility function v_A, will be said more prudent than Agent B, with utility function v_B, if $\psi(x, \tilde{e}, w_A) \geqslant \psi(x, \tilde{e}, w_B)$ for all $(x, \tilde{e})$, where $w_A \equiv \nabla v_A \cdot \rho$ and $w_B \equiv \nabla v_B \cdot \rho$.

Since w is continuous and monotone in the first variable, we may define the $(x_1, \ldots, x_n)$-inverse of w, $w^{-1}(\cdot, x_1, \ldots, x_n)$, as follows: $w(x_0, x_1, \ldots, x_n) = y \Leftrightarrow w^{-1}(y, x_1, \ldots, x_n) = x_0$. We obtain the following characterization of the fact that Agent A is more prudent than Agent B.

Proposition 3. *The following three conditions are equivalent*:

1. *Agent A is more prudent than Agent B, i.e., $\psi(x, \tilde{e}, w_A) \geqslant \psi(x, \tilde{e}, w_B)$ for all $(x, \tilde{e})$.*
2. *The function $\varphi(y, x_1, \ldots, x_n) \equiv w_A[w_B^{-1}(y, x_1, \ldots, x_n), x_1, \ldots, x_n]$ is convex in $(y, x_1, \ldots, x_n)$.*
3. *The matrix $R(x) \equiv [P^{w_A} - P^{w_B}](x)$ is positive semidefinite for all x.*

[5] The set of utility functions that satisfy the conditions in Proposition 2 includes the class of power functions of the form $v(x_0, x_1, \ldots, x_n) = -x_0^{\alpha_0} \ldots x_n^{\alpha^n}$ with $\alpha_i \leqslant 0$ and the class of exponential functions of the form $v(x_0, \ldots, x_n) = -\exp(\alpha_0 x_0 + \cdots + \alpha_n x_n)$ with $\alpha_i \leqslant 0$.

Proof. The proof follows the lines of the proof of Theorem 3 in Karni [14] replacing the (indirect) utility function v by the negative of the marginal utility of saving, $-w$. □

We conclude that the intensity of the precautionary saving motive in a multivariate setting is unambiguously captured by our matrix-measure of multivariate prudence P^w, for both small and large multidimensional risks.[6]

2.4. The Drèze–Modigliani substitution effect in a multivariate setting

In the classical setting of precautionary saving with a univariate risk, Drèze and Modigliani [8] established the presence of a positive substitution effect, i.e. that under decreasing absolute risk aversion, the individual will find it optimal to save more than the compensation necessary to be indifferent to the presence of the risk. Kimball [16] interpreted this positive substitution effect as "a simple consequence of the fact that absolute prudence is greater than absolute risk aversion when absolute risk aversion is decreasing". In the next, we analyze to which extent this result pertains in a multivariate setting.

We may measure the impact of a shift in the direction of ρ by considering the function Π^ρ defined by $\Pi^\rho(x,\tilde{e},v,s) \equiv \Pi(x+s\rho,\tilde{e},v)$, where $\Pi(x,\tilde{e},v)$ is the risk premium, as defined[7] by Karni [14]. We will say that the risk premium is decreasing in the direction of ρ if $\frac{\partial}{\partial s}\Pi^\rho(x,\tilde{e},v,s) < 0$. This concept corresponds to the concept of endogenously decreasing risk aversion of Drèze and Modigliani [8]. By definition, we have

$$E\big[v(\tilde{x}+s\rho)\big] = v\big(x_0+\rho_0 s - \Pi^\rho(x,\tilde{e},v,s), x_1+\rho_1 s,\dots,x_n+\rho_n s\big). \tag{4}$$

Differentiating this expression with respect to s, we easily obtain that the risk premium is decreasing in the direction of ρ if and only if we have

$$E\big[w(\tilde{x})\big] > w\big(x_0 - \Pi(x,\tilde{e},v), x_1,\dots,x_n\big).$$

By definition, this inequality becomes an equality when replacing $\Pi(x,\tilde{e},v)$ by the equivalent precautionary premium $\psi(x,\tilde{e},v)$. Given the concavity of v in x_0, we obtain the following result.

Proposition 4. *The following conditions are equivalent*:

1. *The risk premium is decreasing* (*resp. nonincreasing*[8]) *in the direction of* ρ.
2. *The precautionary premium is larger than* (*resp. larger than or equal*) *the risk premium.*
3. *The matrix* $K(x) \equiv [P^w - A^v](x)$, *with* $P^w \equiv [\frac{-w_{ij}}{w_0}]$ *and* $A^v \equiv [\frac{-v_{ij}}{v_0}]$, *is positive definite* (*resp. semidefinite*) *for all* x.

Proof. The equivalence of 1. and 2. follows from the discussion above. To show the equivalence of 3. and 1. notice that the matrix P^w can be written as $P^w = A^v + [\frac{\partial}{\partial s}A^{v,\rho}(0)]\frac{v_0}{w_0}$ where $A^{v,\rho}(s) \equiv A(x+s\rho)$. Therefore, $[P^w - A^v] = [\frac{\partial}{\partial s}A^{v,\rho}(0)]\frac{v_0}{w_0}$. Note that $A^{v,\rho} = A^{v^{\rho,s}}$

[6] To illustrate, $v_A(x_0,\dots,x_n) = -\exp(\gamma\alpha_0 x_0 + \cdots + \gamma\alpha_n x_n)$, with $\alpha_i \leqslant 0$ and $\gamma \geqslant 1$, is more multivariate prudent than $v_B(x_0,\dots,x_n) = -\exp(\alpha_0 x_1 + \cdots + \alpha_n x_n)$.

[7] See Footnote 3.

[8] The concept of nonincreasing risk premium is the natural extension of the previously defined concept of decreasing risk premium.

where $v^{\rho,s}(x) \equiv v(x + s\rho)$. Given $w_0 < 0$, $[P^w - A^v]$ is positive definite if and only if $[\frac{\partial}{\partial s} A^{\rho,v}(0)]$ is positive definite or equivalently if and only if $A^{v^{\rho,s}} - A^v$ is negative definite for all $s > 0$. Following Karni [14] this means that $v^{\rho,s}$ is less risk averse than v or equivalently that $\Pi(x, \tilde{e}, v^{\rho,s}) < \Pi(x, \tilde{e}, v)$. It suffices to remark that $\Pi(x, \tilde{e}, v^{\rho,s}) = \Pi((x + s\rho), \tilde{e}, v)$ to see that the condition $\Pi(x, \tilde{e}, v^{\rho,s}) < \Pi(x, \tilde{e}, v)$ for all $s > 0$ is equivalent to the decrease of $\Pi(x, \tilde{e}, v)$ in the direction of ρ. □

In other words, multivariate prudence is stronger than multivariate risk aversion. Of course, if the risk premium is increasing (resp. constant), then the precautionary premium is smaller than (resp. equal to) the risk premium.[9] This result can be seen as a natural extension to the multidimensional setting of the results of Drèze and Modigliani [8] and Kimball [16] on the substitution effect.

3. Multivariate downside risk aversion

3.1. Precautionary saving and endogenous harm disaggregation

In the context of a single-attribute utility function, precautionary saving has been linked with a preference towards disaggregation of harms known as downside risk aversion [20]. Downside risk aversion is defined by Eeckhoudt and Schlesinger [9] as follows: given a certain reduction in wealth $-k$ and a mean-zero random variable $\tilde{e}$, an individual is downside risk averse if he prefers the lottery $[x - k; x + \tilde{e}]$ to the lottery $[x; x - k + \tilde{e}]$, where the outcomes occur with equal probability. In other words, the individual perceives the harms $\tilde{e}$ and $-k$ as 'mutually aggravating' [17]. In an expected utility framework, downside risk aversion is equivalent to a positive third derivative of the utility function.

Eeckhoudt et al. [10] extended these ideas to a bivariate setting by defining the concept of cross-prudence. Given two attributes, say wealth x_0 and health x_1, an individual is said cross-prudent (in wealth) if he prefers the lottery $[x_0 - k, x_1; x_0, x_1 + \tilde{e}]$ to the lottery $[x_0, x_1; x_0 - k, x_1 + \tilde{e}]$. In an expected utility framework, cross-prudence is equivalent to the marginal utility of wealth being convex in health.

To extend these ideas to a framework with general multivariate risks, and to relate them to our previously introduced notion of multivariate prudence, we propose the following definition.

Definition 3. An individual displays Multivariate Downside Risk Aversion (MDRA) – in the direction of ρ – if for all $k \in \mathbb{R}_+^*$, for all $x \in \mathbb{R}^{n+1}$ and all multivariate risk $\tilde{e}$,

$$[x - k\rho; \tilde{x}] \succ [x; \tilde{x} - k\rho]$$

where, in each lottery, the outcomes have equal probability and $\tilde{x} \equiv x + \tilde{e}$.

As in the univariate or bivariate settings, the intuition for this definition is that an individual prefers to locate a harm in the form of a multivariate risk to states of nature in which the value of the attributes is relatively high. When all but one of the attributes are held constant in all states of nature, our definition corresponds to the univariate concept of downside risk aversion.

[9] For example, the exponential utility function $v(x_0, \ldots, x_n) = -\exp(\alpha_0 x_0 + \cdots + \alpha_n x_n)$, with $\alpha_i \leqslant 0$, exhibits a constant risk premium.

When there is a single mean-zero risk in a given attribute, a single certain reduction in a different attribute, and all other attributes are held constant, our definition corresponds to Eeckhoudt et al.'s [10] concept of cross-prudence.

In the expected utility framework, an individual, endowed with a vNM utility function v of class C^2 displays MDRA (in the direction of ρ) if the following condition holds

$$\frac{1}{2}v(x-k\rho)+\frac{1}{2}E\big[v(\tilde{x})\big]>\frac{1}{2}v(x)+\frac{1}{2}E\big[v(\tilde{x}-k\rho)\big] \quad \text{for all } (x,k,\tilde{e}).$$

The following proposition establishes the condition for preferences to display MDRA.

Proposition 5. *In an expected utility framework, an individual, endowed with a utility function v of class C^2, displays MDRA (in the direction of ρ) if and only if the function $w \equiv \nabla v \cdot \rho$ is convex.*

Proof. For the first implication, we have $E[v(\tilde{x})]-E[v(\tilde{x}-k\rho)]>v(x)-v(x-k\rho)$. Since $k\in\mathbb{R}_+^*$ can be made arbitrarily small, we get $E[w(\tilde{x})]>w(x)$. By Jensen's inequality, this implies that w is convex.

Let us prove the reverse implication. By Jensen's inequality, we have $E[w(\tilde{z})]>w(z)$ for all random variable $\tilde{z}$ such that $E[\tilde{z}]=z$. This implies that for all $h>0$, $E[w(\tilde{x}-h\rho)]>w(x-h\rho)$, hence for all $k>0$, $\int_0^k E[w(\tilde{x}-h\rho)]\,dh>\int_0^k w(x-h\rho)\,dh$. We then get $E[v(\tilde{x})]-E[v(\tilde{x}-k\rho)]>v(x)-v(x-k\rho)$, which implies that the individual displays MDRA. □

Now recall that convexity of the marginal utility of saving w in X is also a necessary and sufficient condition for multivariate prudence (Proposition 2). Being *multivariate downside risk averse* in the direction of ρ is equivalent to being *multivariate prudent* in the sense of the previous section. Therefore, our definition of MDRA is equivalent to a positive precautionary savings motive.

3.2. The intensity of MDRA

Despite the equivalence between multivariate prudence and MDRA, their intensity need not be measured in the same way.[10] To evaluate the intensity of MDRA we consider, as Crainich and Eeckhoudt [4] do in a univariate framework, the certain amount of income $\theta(x,\tilde{e},v,k)$ that compensates for the difference in expected utility, i.e., such that

$$\frac{1}{2}v(x-k\rho)+\frac{1}{2}E\big[v(\tilde{x})\big]=\frac{1}{2}v(x+\theta\delta)+\frac{1}{2}E\big[v(\tilde{x}-k\rho)\big]. \tag{5}$$

A second order approximation of both sides of (5) yields

$$v(x)+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}v_{ij}(x)\sigma_{ij}\approx v(x+\theta\delta)+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}v_{ij}(x-k\rho)\sigma_{ij}.$$

Using a first order approximation for small k and small θ, we get

[10] This has been emphasized in the univariate framework by a number of authors [4,6,13,21].

$$\theta(x, \tilde{e}, v, k) \approx \frac{k}{2} tr\big[V^e D^v(x)\big], \tag{6}$$

where D^v denotes the matrix $D^v \equiv [\frac{w_{ij}}{v_0}]$.

D^v is our proposed matrix-measure of local MDRA. We have $D^v = (-\frac{w_0}{v_0})P^w$. The following proposition establishes a global property of D^v relative to θ.

Proposition 6. *The following conditions are equivalent.*

1. *The compensating amount is nonnegative, $\theta(x, \tilde{e}, v, k) \geqslant 0$ for all $(x, \tilde{e}, k)$.*
2. *The matrix $D^v(x)$ is positive semidefinite for all x.*

Proof. Since v is nondecreasing, the amount $\theta(x, \tilde{e}, v, k)$ is nonnegative if and only if v displays MDRA. By Proposition 2, we know that it is equivalent to the convexity of w. The function w is convex if and only if the matrix $[w_{ij}]$ is positive semidefinite. Since $v_0 > 0$, this is equivalent to D^v positive semidefinite. □

A natural question that arises then is: can we compare the MDRA attitude of two agents as we did with the measure of multivariate prudence? To answer this question we begin with the following definition.

Definition 4. Agent A, with utility function v_A, is said more multivariate downside risk averse than Agent B, with utility function v_B, when $\theta(x, \tilde{e}, v_A, k) \geqslant \theta(x, \tilde{e}, v_B, k)$ for all $(x, \tilde{e}, k)$.

We obtain the following local result,[11]

Proposition 7. *In the case of small risks, $\theta(x, \tilde{e}, v_A, k) \geqslant \theta(x, \tilde{e}, v_B, k)$ for all $(x, \tilde{e}, k)$ if and only if the matrix $[D^{v_A} - D^{v_B}](x)$ is positive semidefinite for all x.*

Proof. Suppose first that the matrix $[D^{v_A} - D^{v_B}](x)$ is positive semidefinite. In the case of small risks, we have $\theta(x, \tilde{e}, v_A, k) - \theta(x, \tilde{e}, v_B, k) \approx \frac{k}{2} tr[V^e(D^{v_A} - D^{v_B})(x)]$. Suppose now that $\theta(x, \tilde{e}, v_A, k) \geqslant \theta(x, \tilde{e}, v_B, k)$. We then have

$$\frac{k}{2} tr\big[V\big(D^{v_A} - D^{v_B}\big)(x)\big] \geqslant 0$$

for all positive semidefinite symmetric matrix V. Consider for a given vector U, the matrix $V \equiv UU^t$, then $U^t(D^{v_A} - D^{v_B})U = tr[U^t(D^{v_A} - D^{v_B})U] = tr[V(D^{v_A} - D^{v_B})] \geqslant 0$, hence $[D^{v_A} - D^{v_B}](x)$ is positive semidefinite. □

4. An application

We conclude our analysis with a simple model of social discounting which permits to establish a closer connection between the proposed measures of MDRA and of multivariate prudence.[12]

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[11] Unfortunately, we have not been able to establish a global condition linking $\theta(x, \tilde{e}, v_A, k)$ and $\theta(x, \tilde{e}, v_B, k)$ with the matrix $[D^{v_A} - D^{v_B}](x)$ for all risks. This, however, is not very surprising given the difficulty in finding global conditions in the univariate case (e.g. [15]).

[12] Crainich and Eeckhoudt [4] perform a similar analysis in the context of the classical problem of social discounting with a single variable.

Suppose that there are two dates and that the welfare of the representative consumer is characterized by a discounted intertemporal expected utility function $H(x_0, \tilde{x}_1) = u(x_0) + E[e^{-\lambda} v(\tilde{x}_1)]$, where λ captures the pure rate of time preference.[13] Following Gollier [11], one can define n social discount rates, one for each good. We will focus on good 0, which, as before, represents the income. To define the social discount rate imagine a marginal project that decreases current income by a sure (small) amount ε and increases income at date 1 by εe^r. Implementing this project is socially efficient if the net trade-off is positive. This rule can be written as $r \geqslant \hat{r} \equiv \lambda - \ln E[v_0(\tilde{x}_1)/u_0(x_0)]$. This threshold $\hat{r}$ is called social discount rate (for good 0). For expositional clarity we will focus on the case $v = u$ and $x_1 = x_0 = x$.

Clearly, the introduction of a multidimensional risk makes the social discount rate lower if social preferences display multivariate prudence or, equivalently, MDRA. To have a sense of the intensity of these attitudes, we can ask two related questions. First, what is the certain reduction in second period income that has the same impact on the social discount rate as the introduction of the multidimensional risk, that is, what is the amount κ such that $E[v_0(\tilde{x})] = v_0(x - \kappa\delta)$? Clearly, such amount equals the equivalent precautionary premium, i.e. $\kappa = \Psi(x, \tilde{e}, v_0)$. As we have seen in Section 2, for small risks we have $\Psi(x, \tilde{e}, v_0) \approx \frac{1}{2} tr[V^e P^{v_0}(x)]$. Alternatively, we can ask: what is the reduction in the rate of time preference, say $\Delta\lambda$, that has the same impact on the social discount rate as the introduction of the multidimensional risk (i.e. such that $\hat{r} = \lambda - \Delta\lambda + \ln[v_0(x_1)/v_0(x_0)]$)? For small risks the answer is $\Delta\lambda \approx \frac{1}{2} tr[V^e D^v(x)] = \theta(x, \tilde{e}, v)$.

In other words, the social discount rate in an economy with uncertain endowments $\tilde{x}_1$ and a rate of time preference λ is the same as the discount rate in an economy with certain endowments $x - \Psi(x, \tilde{e}, v_0)\delta$ and a rate of time preference λ, and it is also the same as the discount rate in an economy with certain endowments x and a rate of time preference $\lambda - \theta(x, \tilde{e}, v)$. The matrix-measures P^{v_0} and D^v capture, respectively, the magnitude of the equivalent compensations.

More generally, let us now consider a marginal project that decreases current income by a sure small amount ε and increases future consumption by $\varepsilon e^r \rho$ (e.g. an investment in a clean technology which potentially increases future income, environmental quality, and health). The minimum threshold that the rate of return r must exceed for the project to be socially efficient is then given by $\hat{r} = \lambda - \ln(E[w(\tilde{x})]/v_0(x))$. The certain reduction in date 1 income κ that has the same impact on the interest rate as the introduction of the multidimensional risk is then such that $E[w(\tilde{x})] = w(x - \kappa\delta)$ and κ again corresponds to the equivalent precautionary income premium, i.e. $\kappa = \Psi(x, \tilde{e}, w)$. On the other hand, the reduction in the rate of time preference that has the same impact on the social discount rate as the introduction of the multidimensional risk is given by $\Delta\lambda = \theta(x, \tilde{e}, v)[\frac{w(x)}{v_0(x)}]^{-1}$. Note that $\frac{w(x)}{v_0(x)}$ measures the marginal impact of ρ in monetary units.[14] Therefore, if we normalize this impact to equal one unit of income, as is commonly done in cost-benefit analyses, we have that $\frac{w(x)}{v_0(x)} = 1$ and $\Delta\lambda = \theta(x, \tilde{e}, v)$.

5. Conclusion

The theory of precautionary saving and the measurement of the strength of the precautionary saving motive have been topics of extensive research. In a review of the literature Carroll

13 Our analysis is closely related to that of Gollier [11], who evaluated the problem of social discounting in a framework where the representative consumer has a utility function defined over two attributes, income/consumption and environmental quality, which evolve stochastically over time.

14 That is, $\frac{w(x)}{v_0(x)} = \sum_{i=0}^{n} \rho_i \frac{v_i(x)}{v_0(x)}$, where $\frac{v_i(x)}{v_0(x)}$ is the relative price of attribute i.

and Kimball [2] conclude that “The qualitative and quantitative aspects of the theory of precautionary behavior are now well established”. Although this is certainly the case in the context of a single-attribute utility function and a single source of risk much less is known about precautionary behavior in the presence of a multidimensional risk. The objective of this paper has been to start filling this gap in the literature. We derived a matrix-measure of multivariate prudence and we showed that, in the presence of a multidimensional risk, this measure is useful for comparing precautionary behavior among individuals and for comparing the strength of precautionary saving relative to the strength of risk aversion (i.e. the Drèze–Modigliani substitution effect).

Extending the work of Eeckhoudt and Schlesinger [9] and Eeckhoudt et al. [10], we also showed that precautionary saving behavior in the presence of a multidimensional risk is closely related with a preference towards lotteries in which a multidimensional risk is present when the level of the attributes is low rather than when the level of the attributes is high. We proposed an alternative measure that captures the strength of such a preference and we presented an application showing that both of our measures may be useful for evaluating different aspects of economic phenomena.

Our results open the way for empirical tests of multivariate attitudes towards risk. The precautionary premium can be measured by comparing optimal saving levels with and without noise. Once the precautionary premium is observed for different structures of noise, the prudence matrix can be determined through Proposition 1. Finally, bootstrap methods can be used to determine if the prudence matrix is positive semidefinite (e.g. [12]) in order to determine which categories of agents exhibit prudence (Proposition 2). An experimental setting along the lines of Deck and Shlesinger [5] would provide a complementary approach to test for the presence of MDRA or equivalently multivariate prudence (Proposition 5).

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