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Equilibrium Pricing in Incomplete Markets

Abdelhamid Bizid and Elyès Jouini*

Abstract

Given the exogenous price process of some assets, we constrain the price process of other assets that are characterized by their final payoffs. We deal with an incomplete market framework in a discrete-time model and assume the existence of the equilibrium. In this setup, we derive restrictions on the state-price deflators. These restrictions do not depend on a particular choice of utility function. We investigate numerically a stochastic volatility model as an example. Our approach leads to an interval of admissible prices that is more robust than the arbitrage pricing interval.

I. Introduction

The pricing of contingent claims is rooted in the pioneering work of Black and Scholes (1973) and Harrison and Kreps (1979). Their results are based on the key idea that the prices of existing assets induce a unique arbitrage-free price for any new redundant asset. For a new asset that is not redundant, however, the price must lie in an arbitrage-free interval. A large part of the literature deals with the reduction of this interval in order to restrict the bid-ask spread and to obtain an unambiguous price.

In our paper, we propose a new approach that consists of exploiting the partial conditions that arise from equilibrium analysis. We assume that the price processes of some assets are given exogenously (e.g., follow from econometric estimations) and we want to constrain the price processes of other assets (typically, derivative assets) that are characterized by their value at maturity. Dealing with an a priori incomplete market framework, we obtain bounds on the density of the pricing probabilities and, thereafter, on the asset prices.

Our approach is related to the payoff distribution pricing model (PDPM) introduced by Dybvig (1988a), (1988b), who considers the implications of the individual optimality conditions on the asset price for agents with Von Neumann-Morgenstern (VNM) preferences, and derives conclusions in terms of ordering.

*Bizid, abdelhamid.bizid@rabobank.com, Rabo Securities, Fund Derivatives Structuring, Thames Court, One Queenhithe, London, EC4V 3RL, U.K., and Jouini, jouini@ceremade.dauphine.fr, Université Paris 9-Dauphine and Institut Universitaire de France, Place du Maréchal de Lattre de Tassigny, 75 775 Paris Cedex 16, France. The authors thank Gurdip Bakshi (associate editor and referee) for helpful comments. This article is a modified version of a previous work by the authors and does not in any way reflect the views of Rabo Securities N.V.

This model is extended to a friction market case (including that of incomplete markets) by Jouini and Kallal (2001), who show that this approach does not permit a reduction of the interval of admissible prices. The main innovation with the present work is that we explicitly use the market clearing conditions.

Perrakis and Ryan (1984) and Perrakis (1986) both present restrictions on the family of probabilities used to price derivative assets, without specifying an economic model. Perrakis (1986) discusses the restriction for a single-period model only. In Perrakis and Ryan (1984), the restriction is taken as a primitive assumption in a dynamic model in order to obtain bounds on European and American options. In fact, the authors propose an ordering principle on the risk-neutral probabilities. Bizid, Jouini, and Koehl (1999) establish that the underlying economic assumption of the Perrakis (1986) and Perrakis and Ryan (1984) papers is, in fact, that the market model is complete and further that at each node the different transitions are equiprobable. It is easy to adapt their result to situations where the transitions are not equiprobable. However, applying Perrakis and Ryan's (1984) computational method to an incomplete market framework amounts to restricting one's attention to equilibrium prices that are compatible with at least one completion of the market. Contrary to the intuition given by the optimal investment-consumption problem (see, e.g., Karatzas and Shreve (1998)), the pricing interval that is consistent with an equilibrium in an incomplete framework is wider than the one that is consistent with at least one completion. Therefore, from an equilibrium point of view, it appears that the asset pricing problem in an incomplete market is neither equivalent to the same problem in a complete market with incomplete data nor to an incomplete market with a zero potential demand in the risk directions that are orthogonal to the market. The specific approach we develop in this paper for the incomplete market setting seems then to be a worthy contribution.

The paper proceeds as follows. Section II presents a simple example that shows the specificity of our approach. Then, we state the discrete-time dynamic model and the associated equilibrium restrictions in Section III. In this model, there are $m + 1$ assets, namely, one equity claim with a total supply normalized to one and m purely financial assets, each one with a total supply equal to zero. We can generalize this setting to a multiasset market in which the claim is given by the market index and our methodology applies to any asset (in zero or nonzero total supply) defined by its terminal payoff. Note also that the derivative assets are not specified (they can depend on a projection of the claim on some source of uncertainty or they can even be uncorrelated with it). Furthermore, the derivative assets are not supposed to complete the market, as the degree of incompleteness is unknown. Agents have general VNM utility functions. Assuming that the price process of the equity claim is exogenously known, we derive equilibrium restrictions on the state-price deflators. In Section IV, we take the price processes of the equity claim and of some purely financial assets as exogenous. As a special case, we consider the existence of a risk-free asset. We state our main result in terms of risk-neutral probabilities and show that we can narrow the admissible pricing interval if the information set is enlarged to other assets. In Section V, we investigate numerically a model with uncertain volatility and we give various examples. The reduction of the bid-ask spread with respect to the arbitrage pricing interval (i.e., the interval compatible with the no-arbitrage condition) motivates the interest of our approach.

II. A Simple Example

To explain our method of pricing, and before introducing our model, we consider the simplest example of incomplete markets with three assets: a quadrinomial one-period model where the sample space and probability are, respectively, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $P = (1/4, 1/4, 1/4, 1/4)$. There are two assets in the market whose price processes are known. The first is a primitive asset whose prices are $p(0) = 25$ at date 0 and $p(\omega_1) = 20$, $p(\omega_2) = 30$, $p(\omega_3) = 40$, and $p(\omega_4) = 50$ at date 1. The second is a risk-free asset defined by interest rate $r = 1/19$.

By definition, risk-neutral probabilities satisfy

$$\frac{1}{1+r}\sum_{i=1}^{4} q(\omega_i)\, p(\omega_i) = p(0) \qquad \text{and} \qquad \sum_{i=1}^{4} q(\omega_i) = 1,$$

where $q(\omega_i)$ are positive for every i. After simple calculations, we obtain $q(\omega_1) = 7/19 + q(\omega_3) + 2q(\omega_4)$ and $q(\omega_2) = 12/19 - 2q(\omega_3) - 3q(\omega_4)$.

The feasible risk-neutral probabilities are then in an area delimited by three points (in the unit simplex),

$$\left(\frac{7}{19}, \frac{12}{19}, 0, 0\right), \quad \left(\frac{15}{19}, 0, 0, \frac{4}{19}\right), \quad \text{and} \quad \left(\frac{13}{19}, 0, \frac{6}{19}, 0\right).$$

We now consider a completion of this market with non-redundant derivative assets in zero net supply. There exists then a representative agent in this economy with utility function u. Moreover, if there is one unit of primitive asset and no other endowment in the economy, the total wealth W (of the representative agent) is equal to p.

At the equilibrium, the maximization program of this agent leads to the following first-order conditions,

$$q(\omega_1) = \frac{u'(20)}{u'(25)}, \quad q(\omega_2) = \frac{u'(30)}{u'(25)},$$
$$q(\omega_3) = \frac{u'(40)}{u'(25)}, \quad q(\omega_4) = \frac{u'(50)}{u'(25)}.$$

Since u is strictly concave, we obtain the following relationship, which is put forward by Perrakis (1986), $q(\omega_1) > q(\omega_2) > q(\omega_3) > q(\omega_4)$. After imposing these restrictions, we get the feasible range of the risk-neutral probability delimited by the four following points,[1]

$$\begin{aligned} Q_1 &\equiv \left(\frac{11}{19}, \frac{4}{19}, \frac{4}{19}, 0\right), \\ Q_2 &\equiv \left(\frac{26}{57}, \frac{26}{57}, \frac{5}{57}, 0\right), \\ Q_3 &\equiv \left(\frac{12}{19}, \frac{2}{19}, \frac{2}{19}, \frac{2}{19}\right), \quad \text{and} \\ Q_4 &\equiv \left(\frac{71}{152}, \frac{71}{152}, \frac{5}{152}, \frac{5}{152}\right). \end{aligned}$$

Note that this range is smaller than that in the unrestricted case. Furthermore, we obtain these smaller bounds under an existence assumption about the derivative assets that completes the market without specifying the price process of the assets. Since we deal with a linear pricing rule, the upper and lower bound on the equilibrium price of a given derivative asset defined by its payoff $x=(x_1, x_2, x_3, x_4)$ is reached at one of these four extremal points. We then have the following bounds,

$$\left[\inf_{Q_i} E^{Q_i}(x) \ ; \ \sup_{Q_i} E^{Q_i}(x)\right].$$

Consequently, this approach permits one to obtain bounds on the price of the derivative asset that are better than the usual arbitrage-free bounds.

Now if we consider our market directly and do not assume in this case that the market is completed by some assets, we cannot apply the representative agent technique. However, we can find necessary conditions in the equilibrium on the risk-neutral probabilities. Using the maximization program of the nth agent, the density of his pricing probability with respect to P must satisfy

$$\begin{aligned} q^n(\omega_1) &= \frac{u_n'(C^n(\omega_1))}{u_n'(C^n(0))}, \quad q^n(\omega_2) = \frac{u_n'(C^n(\omega_2))}{u_n'(C^n(0))}, \\ q^n(\omega_3) &= \frac{u_n'(C^n(\omega_3))}{u_n'(C^n(0))}, \quad q^n(\omega_4) = \frac{u_n'(C^n(\omega_4))}{u_n'(C^n(0))}, \end{aligned}$$

where $C^n(\omega)$ denotes the optimal consumption of agent n at the state of the world ω. At date 1, we have $p(\omega_i) = \sum_n C^n(\omega_i)$. Then, since $p(\omega_1) < p(\omega_2)$, there exists at least one agent n_0 such that $C^{n_0}(\omega_1) < C^{n_0}(\omega_2)$. Since u_{n_0} is strictly concave, we obtain that for agent n_0, $q^{n_0}(\omega_1) > q^{n_0}(\omega_2)$. This restriction and the

[1]Graphically, it suffices to consider the set of all feasible risk-neutral probabilities previously defined and to consider the area delimited by the constraints $q(\omega_i) > q(\omega_j)$ and $i < j$ as we do in Figure 1.

martingale measure condition lead to the fact that the feasible range of pricing probabilities must lie in the area delimited by the following four points,[2]

$$\begin{aligned} Q_1^{1,2} &\equiv \left(\frac{13}{19}, 0, \frac{6}{19}, 0\right), \\ Q_2^{1,2} &\equiv \left(\frac{26}{57}, \frac{26}{57}, \frac{5}{57}, 0\right), \\ Q_3^{1,2} &\equiv \left(\frac{15}{19}, 0, 0, \frac{4}{19}\right), \quad \text{and} \\ Q_4^{1,2} &\equiv \left(\frac{9}{19}, \frac{9}{19}, 0, \frac{1}{19}\right). \end{aligned}$$

The equilibrium price of a given derivative must then belong to

$$\left[\inf_{Q_i^{1,2}} E^{Q_i^{1,2}}(x) \ ; \ \sup_{Q_i^{1,2}} E^{Q_i^{1,2}}(x)\right].$$

Using the same methodology for each pair (i,j) such that $i<j$, we determine that there exist five agents n_1, n_2, n_3, n_4, and n_5 such that

$$\begin{aligned} C^{n_1}(\omega_1) &< C^{n_1}(\omega_3), \qquad & C^{n_2}(\omega_2) &< C^{n_2}(\omega_3), \\ C^{n_3}(\omega_1) &< C^{n_3}(\omega_4), \qquad & C^{n_4}(\omega_2) &< C^{n_4}(\omega_4), \qquad \text{and} \\ C^{n_5}(\omega_3) &< C^{n_5}(\omega_4), \end{aligned}$$

leading to five other pricing intervals. Therefore the admissible interval of prices is necessarily the intersection of the six pricing intervals. Figure 1 represents the no-arbitrage probability set (the big triangle) and the different more restrictive sets (associated with each partial order). The intersection of these sets is the probability set for a complete market framework (Perrakis' (1986) assumption), as there is in fact a global order on the pricing probabilities.

As an example, we price a call option with a strike equal to 35. The classical no-arbitrage argument gives the following pricing bounds $[0\,;3.16]$ at $t=0$. Perrakis' (1986) bound is $[0.44\,;2.11]$. The incomplete market framework gives the following bound $[0.44\,;2.37]$ at date-$t=0$. Therefore, even in an incomplete framework, we can obtain restrictions on option prices that are distinct from—but quite close to—the complete market framework. From a mathematical point of view, if we denote by T_i, $i=1,\ldots,6$, the six different sets, the Perrakis' bound is given by

$$\left[\inf_{\cap T_i} E(x) \ ; \ \sup_{\cap T_i} E(x)\right],$$

and our bound is given by

$$\cap \left[\inf_{T_i} E(x) \ ; \ \sup_{T_i} E(x)\right].$$

[2]As before, it suffices to consider the set of all feasible risk-neutral probabilities and to consider the area delimited by the constraint $q(\omega_1) > q(\omega_2)$ as we do in Figure 1.

FIGURE 1

Intervals of the Pricing Probabilities

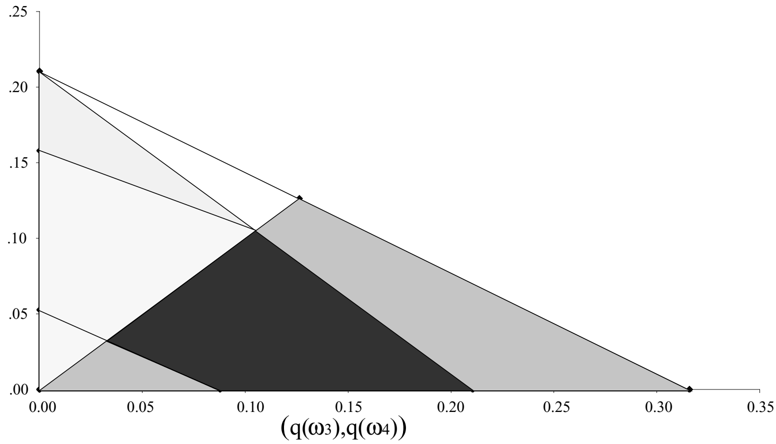

Figure 1 represents the six different sets of pricing probabilities. Note that, since we deal with a quadrinomial framework, these intervals are characterized by the pair $(q(\omega_3), q(\omega_4))$. The largest triangle represents the no-arbitrage interval. The black interval (the intersection of the six intervals) is associated with Perrakis' (1986) assumption.

Even if this definition looks like the previous one, these sets are not equal unless the market is assumed to be complete. In such a situation, there exists an agent that satisfies all the consumption constraints; in fact, all the agents do. In other words, it is possible to take $n_0 = n_1 = \cdots = n_5$.

Next, we consider a multiperiod framework and explain how to use this recursive procedure for pricing derivatives in an incomplete discrete-time market.

III. Framework and Equilibrium Restrictions

In the following, we suppose that there is one non-storable consumption good. For the sake of simplicity, we assume that only one firm produces this good. At each date-t, the firm produces d_t units of the good, which is distributed as dividends to shareholders who own the firm. There is one perfectly divisible equity claim, which is tradable on date-t at the post-dividend price p_t in terms of the consumption good. After date-t, the firm becomes obsolete and is valued at zero. A share θ of this claim ensures that the owner receives quantity θd_t of the perishable good at date-t. Throughout the paper, the total supply of the equity claim is normalized to one. Our results do not change if we assume the existence of more than one productive asset. In that case, it would suffice to replace the process d_t by the total dividend process and to assume in the numerical examples (Section V) that this total dividend and the total wealth (instead of p_t) are similarly ordered. Furthermore, when there is more than one asset, our approach provides pricing bounds for productive assets as well as for derivative assets on any underlying asset, and not only on the index. More precisely, with our method it suffices to observe the index process and terminal payoff of any asset in order to derive pricing bounds on this asset.

In addition to the equity claim described above, there are m purely financial assets (i.e., assets with total supply always equal to zero). For $i=1,\ldots,m$, the ith financial asset is tradable at each date at the price q_t^i in terms of the consumption good.

Formally, we consider a model with a finite number of states and dates, where all random processes share a common filtered probability space in which P is the (true) probability and $E_t\left[.\right]$ denotes the expectation conditional on what is known at t. We denote by Σ_t the set of all date-t nodes and, for $\sigma_t \in \Sigma_t$, $f\left(\sigma_t\right)$ is the set of the immediate successors of the date-t node σ_t.

There are N consumers. The nth consumer has a VNM utility function $U^n\left(.\right)$, which generates for any consumption process $\left(C_t\right)_{0\leq t\leq T}$, the following utility level at date 0,

$$U^n\left(C\right) \quad = \quad E\left[\sum_{t=0}^{T} u^n\left(C_t, t\right)\right],$$

where u^n maps $\mathbb{R}^{+*} \times \{0,\ldots,T\}$ in $\mathbb{R}$. We assume the following classical properties with respect to u^n. For all n, u^n is continuously differentiable, increasing, and strictly concave. Moreover, we impose the following Inada condition,

$$\forall t, \quad t=0,\ldots,T, \quad u^n\left(x\,;t\right) \underset{x\to 0^+}{\to} -\infty.$$

The sequence of events is as follows. First, the firm produces and distributes the dividends among the shareholders; second, consumption, new portfolios, and new prices obtain. The prices come from the equilibrium conditions and, as usual, the agents take the prices as given in their utility maximization program. Trading strategy S is the vector,

$$\left\{\left(C_t\right)_{t=0,\ldots,T}\,;\left(\theta_t\right)_{t=1,\ldots,T}\,;\left(\alpha_t\right)_{t=1,\ldots,T} = \left(\left(\alpha_t^i\right)_{i=1,\ldots,m}\right)_{t=1,\ldots,T}\right\},$$

where $\forall\, t$, $t=0,\ldots,T$, C_t depends on the information available at date-t, and $\forall\, t$, $t=1,\ldots,T$, θ_t, and α_t depend on the information available at date-$t-1$.

We interpret θ_t (α_t^i) as the quantity of the equity claim (ith purely financial asset) owned at date-t by the agent who follows strategy S. Hence, θ_0 and α_0 are the initial quantities of the assets owned by the agent. By convention, $\alpha_{T+1}^i = \theta_{T+1}=0$.

We denote by $\alpha_t'.q_t$ the inner product between α_t and q_t. The budget constraints at dates-$t=0,\ldots,T$ are then

(1) $$C_t + \theta_{t+1}p_t + \alpha_{t+1}'.q_t \quad = \quad \theta_t\left(p_t+d_t\right) + \alpha_t'.q_t \quad = \quad W_t,$$

where W_t is interpreted as the date-t wealth before consumption.

For a given agent n with an initial endowment W_0^n, we can define the convex set of the admissible strategies $\mathcal{A}^n$ as the set of the strategies S that satisfy the budget constraint equation (1) and the consumption constraint $C_t \geq 0$ for all t between 0 and T.

By definition, a state-price deflator is an adapted stochastic process ς so that for all t between 0 and $T-1$, we have

$$(2)\qquad \begin{aligned} p_t &= \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1}\left(p_{t+1}+d_{t+1}\right)\right], \\ q_t &= \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}\right]. \end{aligned}$$

In what follows, we denote by Ξ the set of state-price deflators. This set is non-empty since we assume that an equilibrium exists. It is well known that, at the equilibrium and from the first-order optimal conditions, the process $(\varsigma_t^n)_{t=0,\dots,T}$ is defined for any agent n by $\varsigma_0^n = 1$, and for all t between 0 and $T-1$,

$$(3)\qquad \frac{\varsigma_{t+1}^n}{\varsigma_t^n} = \frac{(u^n)'\left(C_{t+1}^{n,*}, t+1\right)}{(u^n)'\left(C_t^{n,*}, t\right)} > 0$$

is a state-price deflator.

If we suppose that the stochastic evolution of the underlying asset price and the terminal date prices of the purely financial assets (i.e., their payoffs) q_T are determined exogenously, our problem consists of how to use these primitives to restrict the initial prices of the purely financial assets q_0 so that they are compatible with an equilibrium. The classical procedure is to use the set Δ of state-price deflators for the stock price process, i.e., the set of the adapted stochastic processes ς, so that for all t between 0 and $T-1$, we have

$$(4)\qquad p_t = \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1}\left(p_{t+1}+d_{t+1}\right)\right].$$

By a no-arbitrage condition, one must then have for all i between 1 and m,

$$(5)\qquad q_0^i \in \left[\inf_{\varsigma\in\Delta} \frac{1}{\varsigma_0} E\left[\varsigma_T q_T^i\right]; \sup_{\varsigma\in\Delta} \frac{1}{\varsigma_0} E\left[\varsigma_T q_T^i\right]\right].$$

We call this interval the no-arbitrage interval.

For a given node $\sigma_t \in \Sigma_t$ and for every pair (σ', σ'') of successors of σ_t, we say that a given state-price deflator is in reverse order of d_{t+1} on (σ', σ'') if

$$(6)\qquad \left(\varsigma(\sigma') - \varsigma(\sigma'')\right)\left(d_{t+1}(\sigma') - d_{t+1}(\sigma'')\right) \leq 0.$$

Then the following result holds.

Theorem 1. At the equilibrium, for every node and every pair (σ', σ'') of successors of this node, there exists a state-price deflator in reverse order of d_{t+1} on (σ', σ'').

Proof. For any date-t and any $\sigma_t \in \Sigma_t$, the total consumption of the economy is $d_t(\sigma_t)$ (recall that $\sum_{n=1}^N \theta_t^n = 1$ and $\sum_{n=1}^N \alpha_t^n = 0$). Then, for a given pair (σ', σ'') of successors of σ_t such that $d_{t+1}(\sigma') > d_{t+1}(\sigma'')$, there exists at least one agent n_0 such that $C_{t+1}^{n_0,*}(\sigma') \geq C_{t+1}^{n_0,*}(\sigma'')$. Note that n_0 depends on σ_t.

Consider the state-price deflator associated with the n_0th agent, as defined in (3). As u^{n_0} is concave and increasing, we obtain

$$0 \quad < \quad \left(u^{n_0}\right)' \left(C_{t+1}^{n_0,*}\left(\sigma'\right), t+1\right) \quad \leq \quad \left(u^{n_0}\right)' \left(C_{t+1}^{n_0,*}\left(\sigma''\right), t+1\right).$$

Multiplying both terms of the inequality by a positive value that depends only on σ_t, we obtain $\varsigma_{t+1}^{n_0}\left(\sigma'\right) \leq \varsigma_{t+1}^{n_0}\left(\sigma''\right)$. It follows that ς^{n_0} is in reverse order of d_{t+1} on $\left(\sigma', \sigma''\right)$ and is a state-price deflator by construction. □

This result implies that for every pair $\left(\sigma', \sigma''\right)$, there exists a state-price deflator ς that satisfies the ordering property, and that there is no one state-price deflator ς that satisfies the ordering property for any $\left(\sigma', \sigma''\right)$. Of course, if the set of admissible state-price deflators is reduced to one process (complete market at the equilibrium), we return to the special case of Perrakis and Ryan's (1984) ordering property. Recall that we need not observe the price processes that complete the market to obtain this result (see Bizid, Jouini, and Koehl (1999)).

Henceforth, we denote by $\Xi_{(\sigma',\sigma'')}$ the set of all the state-price deflators for the stock price process that are in reverse order of d_{t+1} on $\left(\sigma', \sigma''\right)$.

Theorem 2. Consider a date-t node σ_t, with t in $\{0, \ldots, T-1\}$. Suppose that the price values q_{t+1}^i of the ith financial asset ($i \in \{1, \ldots, m\}$) are known for the successors of σ_t. Then $q_t^i(\sigma_t)$ must lie in the interval,

$$\text{(7)} \qquad \left[\max_{(\sigma',\sigma'')} \inf_{\varsigma \in \Xi_{(\sigma',\sigma'')}} \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}^i\right];\ \min_{(\sigma',\sigma'')} \sup_{\varsigma \in \Xi_{(\sigma',\sigma'')}} \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}^i\right]\right],$$

where the infimum and the supremum are taken over $\Xi_{(\sigma',\sigma'')}$ and the maximum and the minimum are taken over all the pairs of successors of σ_t.

Proof. Let i in $\{1, \ldots, m\}$ and let a node $\sigma_t \in \Sigma_t$. Applying Theorem 1, for a given pair $\left(\sigma', \sigma''\right)$ of successors of σ_t, we obtain a state-price deflator ς in reverse order of d_{t+1} on $\left(\sigma', \sigma''\right)$. Therefore, we must have

$$\text{(8)} \qquad q_t^i\left(\sigma_t\right) \quad \in \quad \left[\inf_{\varsigma \in \Xi_{(\sigma',\sigma'')}} \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}^i\right];\ \sup_{\varsigma \in \Xi_{(\sigma',\sigma'')}} \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}^i\right]\right],$$

where the infimum and the supremum are taken over all the state-price deflators for the stock price process that are in reverse order of d_{t+1} on $\left(\sigma', \sigma''\right)$. Using the same argument for any pair of successors of σ_t, $q_t^i(\sigma_t)$ must lie in

$$\text{(9)} \qquad \bigcap_{(\sigma',\sigma'')} \left[\inf_{\varsigma \in \Xi_{(\sigma',\sigma'')}} \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}^i\right];\ \sup_{\varsigma \in \Xi_{(\sigma',\sigma'')}} \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}^i\right]\right],$$

where the intersection is taken over all pairs of successors of σ_t. This gives the bounds of equation (7) and ends the proof. □

Note that interval (7) is not equal to the interval,

$$\text{(10)} \qquad \left[\inf_{\varsigma \in \cap \Xi_{(\sigma',\sigma'')}} \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}^i\right];\ \sup_{\varsigma \in \cap \Xi_{(\sigma',\sigma'')}} \frac{1}{\varsigma_t} E_t\left[\varsigma_{t+1} q_{t+1}^i\right]\right],$$

where the infimum and the supremum are taken over all the state-price deflators for the stock price process that are in reverse order of d_{t+1} on (σ', σ'') for all pairs of successors of σ_t. Indeed, this last interval (which is Perrakis and Ryan's (1984) interval) is much more restrictive and does not apply in an incomplete market framework. However, we can state the following corollary as in Bizid, Jouini, and Koehl (1999).

Theorem 3. (Perrakis and Ryan (1984)) Consider a date-t node σ_t, for $t \in \{0, \ldots, T-1\}$. Suppose that the price values q^i_{t+1} of the ith financial asset (where $i \in \{1, \ldots, m\}$) are known for the successors of σ_t. If the market is complete after the introduction of the financial assets, then $q^i_t(\sigma_t)$ must lie in the interval (10).

Proof. Fix i in $\{1, \ldots, m\}$ and let a node $\sigma_t \in \Sigma_t$. Applying Theorem 1 for a given pair (σ', σ'') of successors of σ_t, we obtain the existence of a state-price deflator ς in reverse order of d_{t+1} on (σ', σ''). Furthermore, since the market is complete, we know that only one state-price deflator exists. This deflator must then be in reverse order of d_{t+1} on (σ', σ'') for all pairs of successors of σ_t, and $q^i_t(\sigma_t)$ must be in the interval provided in equation (10), where the infimum and the supremum are taken over all the state-price deflators for the stock price process that are in reverse order of d_{t+1} on all pairs of successors of σ_t. □

In an equilibrium framework (with incompleteness), the state-price deflators satisfy $\exists\,(i,j) \in \{1, \ldots, f(\sigma_t)\}^2,\ i < j, \quad \varsigma_{t+1}(i) \leq \varsigma_{t+1}(j)$. It appears then that in the complete market case, it suffices to restrict our attention to state-price densities that are globally in reverse order with respect to d_{t+1}, i.e., $\varsigma_{t+1}(1) \leq \varsigma_{t+1}(2) \leq \ldots \leq \varsigma_{t+1}(f(\sigma_t))$, with $d_{t+1}(1) \geq d_{t+1}(2) \geq \ldots \geq d_{t+1}(f(\sigma_t))$, instead of looking to all the state-price density sets that satisfy the constraint $\varsigma_{t+1}(i) \leq \varsigma_{t+1}(j)$ for some pair (i,j) such that $d_{t+1}(i) \geq d_{t+1}(j)$.

We now go back to the incomplete market case. Since, for each asset $i \in \{1, \ldots, m\}$, the payoff q^i_T is assumed to be known, we can apply Theorem 2. We can now use backward induction, as for instance in Ritchken (1985), to compute an interval in which q^i_0 must lie; we call this interval the equilibrium pricing interval. This result establishes bounds on the equilibrium price of any financial asset, using our only knowledge of the dividend process and the equity claim price process (interpreted as a market index).

IV. Adding Assets to the Information Set

In the previous sections, assuming that the price process of the equity claim is determined exogenously, we find equilibrium restrictions on the price processes of the purely financial assets, given their values at the final date-t. In this section, we suppose that not only is the price process of the equity claim p given exogenously, but also the price processes of the $\overline{m}$ first purely financial assets $\left(q^i\right)_{i=1,\ldots,\overline{m}}$, $\overline{m} < m$. Using the same approach as above, we want to restrict the possible values of $\left(q^i\right)_{i=\overline{m}+1,\ldots,m}$. We are particularly interested in the special case where $\overline{m} = 1$ and where the exogenous price process of the first purely financial asset corresponds to a risk-free asset.

Adapting the results of the previous sections, we define the set of state-price deflators for the known price processes, i.e., we adapt the stochastic processes ς so that, for all t between 0 and $T-1$, we have

$$(11) \qquad p_t = \frac{1}{\varsigma_t} E_t \left[\varsigma_{t+1} \left(p_{t+1} + d_{t+1} \right) \right],$$

and for $i = 1, \ldots, \overline{m}$,

$$(12) \qquad q_t^i = \frac{1}{\varsigma_t} E_t \left[\varsigma_{t+1} q_{t+1}^i \right].$$

Obviously, this set is contained in the one defined by the no-arbitrage condition for the equity claim only. Therefore, for any asset i between $\overline{m}+1$ and m, the arbitrage pricing interval is defined as

$$(13) \qquad q_0^i \in \left[\inf \frac{1}{\varsigma_0} E \left[\varsigma_T q_T^i \right] ; \sup \frac{1}{\varsigma_0} E \left[\varsigma_T q_T^i \right] \right],$$

where the infimum and the supremum are taken over the set of state-price deflators for the known price processes. Using Theorem 1 and the fact that every state-price deflator is a state-price deflator for the known price processes, we can apply the methodology of Theorem 2. It is then possible to restrict the interval over which the price of the $m - \overline{m}$ last purely financial assets must lie. As above, we call this interval the equilibrium pricing interval.

We apply now this result to the special case where $\overline{m} = 1$ and q^1 is a strictly positive predictable process. Then we can define the adapted process $(r_t)_{t=0,\ldots,T-1}$ by $1 + r_t = q_{t+1}^1 / q_t^1$. In other words, q^1 is a risk-free asset, and r_t is the risk-free rate between dates-t and $t+1$.

The main difference between this case and that of the previous section is that we must now work with the risk-neutral probabilities and not with the state-price deflators.

For a given risk-neutral probability $\widehat{P}$, we denote by $\widehat{\pi}(\sigma)$ the transition probability between a given date-t node σ_t and one of its successors. For a given pair (σ', σ'') of successors of σ_t, we say that $\widehat{P}_{(\sigma', \sigma'')}$ is in reverse order of d_{t+1} on (σ', σ'') if

$$\frac{\widehat{\pi}(\sigma'')}{\pi(\sigma'')} \leq \frac{\widehat{\pi}(\sigma')}{\pi(\sigma')}.$$

Theorem 4 is the equivalent of Theorem 2 when there exists a risk-free asset with an exogenously given price process. The proof is very close to that of Theorem 2; therefore, we omit it here.

Theorem 4. Let us assume that the price processes of the equity claim p and a risk-free asset (with a total supply always equal to zero), q^1, are given exogenously.

Define $1 + r_t = q^1_{t+1}/q^1_t$, for $0 \leq t \leq T-1$. Then for all i between two and m, given q^i_{t+1}, we have that $q^i_t(\sigma_t)$ must lie in

$$\left[\max_{(\sigma',\sigma'')} \inf_{\widehat{P}_{(\sigma',\sigma'')}} \frac{1}{1+r_t} E_t^{\widehat{P}_{(\sigma',\sigma'')}} \left[q^i_{t+1}\right] ; \right.$$
$$\left. \min_{(\sigma',\sigma'')} \sup_{\widehat{P}_{(\sigma',\sigma'')}} \frac{1}{1+r_t} E_t^{\widehat{P}_{(\sigma',\sigma'')}} \left[q^i_{t+1}\right] \right],$$

where the infimum and the supremum are taken on the martingale measures $\widehat{P}_{(\sigma',\sigma'')}$ that are in reverse order of d_{t+1} on (σ',σ''), and where the maximum and the minimum are taken over all pairs (σ',σ'') of successors of σ_t.

V. An Example of the Stochastic Volatility Model

In this section, we investigate numerically an example in a quadrinomial framework. We consider as given the prices of the equity claim p and of a risk-free asset q^1 and the final payoffs of the $m-1$ other purely financial assets, q_T. Then, using Theorem 4, we compute the arbitrage pricing and equilibrium pricing intervals for each purely financial asset by backward induction.

We consider an n-time steps lattice. Then, the underlying random process is $\left(p_{kT/n}\right)_{k=0,\ldots,n}$, whose distribution is assumed to be known. The discount rate between two successive dates, rT/n, is assumed to be constant.

We consider the following reconnecting tree structure, which models the evolution of the equity claim. If the tree structure's value at date-t is p_t, then $\{p_t u_1(1+rT/n), p_t u_2(1+rT/n), p_t u_2^{-1}(1+rT/n), p_t u_1^{-1}(1+rT/n)\}$ are the four possible values of $p_{t+T/n} + d_{t+T/n}$ at date-$t+T/n$, with

$$u_1 = e^{\sigma_{\max}\sqrt{\frac{T}{n}}} \quad \text{and} \quad u_2 = e^{\sigma_{\min}\sqrt{\frac{T}{n}}}.$$

The tree structure is stationary. We assume the ordering principle with respect to the price process instead of the dividend process. The true transition probabilities are denoted by $\{\pi_i\}_{i=1,\ldots,4} \in \,]0\,;1[^4$. Let $\{\widehat{\pi}_i\}_{i=1,\ldots,4} \in [0\,;1]^4$, which together satisfy $\sum_{i=1}^4 \widehat{\pi}_i = \sum_{i=1}^4 \widehat{\pi}_i u_i = 1$. The existence of such $\{\widehat{\pi}_i\}_{i=1,\ldots,4}$ is guaranteed by the no-arbitrage conditions.

In the rest of the paper, we explain how to use our method to numerically obtain the equilibrium pricing interval. We also compare this interval with the arbitrage pricing interval and the interval obtained by Perrakis and Ryan's (1984) methodology denoted by Perrakis' (1986) pricing interval.

It is difficult to compute the equilibrium pricing interval when the true transition probabilities are not explicitly given. We now consider two concrete examples of true transition probabilities.

A. Equiprobable States of the World

First, assume that $\forall i = 1, \ldots, 4$, $\pi_i = ¼$. We have to solve six linear optimization problems respectively defined by the conditions,

$$\frac{\widehat{\pi}_1}{\pi_1} \leq \frac{\widehat{\pi}_2}{\pi_2}, \frac{\widehat{\pi}_2}{\pi_2} \leq \frac{\widehat{\pi}_3}{\pi_3}, \frac{\widehat{\pi}_3}{p3} \leq \frac{\widehat{\pi}_4}{\pi_4}, \frac{\widehat{\pi}_1}{\pi_1} \leq \frac{\widehat{\pi}_3}{\pi_3}, \frac{\widehat{\pi}_1}{\pi_1} \leq \frac{\widehat{\pi}_4}{\pi_4}, \quad \text{and} \quad \frac{\widehat{\pi}_2}{\pi_2} \leq \frac{\widehat{\pi}_4}{\pi_4}.$$

Since all the problems are linear, it suffices to determine the extreme points of the polyhedron defined by the constraints and then to maximize on these points.

Table 1 shows price intervals for a European call option that is at-the-money with $K = 100$, $T = 1$, $r = 0\%$, and for varying values of $[\sigma_{\min} ; \sigma_{\max}]$.

TABLE 1

Call Price Bounds for a Varying Range of $[\sigma_{\min}, \sigma_{\max}]$

$[\sigma_{\min} ; \sigma_{\max}]$	Perrakis & Ryan's (1984) Pricing	Equilibrium Pricing	No-Arbitrage Pricing
[05% ; 20%]	[5.7991 ; 5.8264]	[4.0180 ; 6.4859]	[1.9895 ; 7.9457]
[06% ; 19%]	[5.6058 ; 5.6298]	[4.2778 ; 6.1860]	[2.3873 ; 7.5496]
[07% ; 18%]	[5.4367 ; 5.4574]	[4.4887 ; 5.9090]	[2.7851 ; 7.1534]
[08% ; 17%]	[5.2907 ; 5.3083]	[4.6575 ; 5.6527]	[3.1827 ; 6.7569]
[09% ; 16%]	[5.1714 ; 5.1855]	[4.7879 ; 5.4245]	[3.5803 ; 6.3604]
[10% ; 15%]	[5.0800 ; 5.0906]	[4.8842 ; 5.2326]	[3.9778 ; 5.9636]
[11% ; 14%]	[5.0198 ; 5.0265]	[4.9458 ; 5.0899]	[4.3752 ; 5.5667]
[12% ; 13%]	[4.9857 ; 4.9881]	[4.9698 ; 5.0086]	[4.7725 ; 5.1697]

A quadrinomial reconnecting tree that models an equity claim diffusion, with volatility belonging to the range $[\sigma_{\min} ; \sigma_{\max}]$. We assume that the number of time steps of the model is $n = 100$. An important piece of information comes from the true probability distribution. We give results that obtain from our approach and no-arbitrage conditions for a European at-the-money call option, over a range of volatilities. We also provide for comparison similar results from Perrakis and Ryan's (1984) method in an incomplete market. It is clear that our equilibrium bounds are not as precise as those of Perrakis and Ryan (1984), but this is due to the fact that we do not arbitrarily assume the market completeness, i.e., the possibility to be completely vega volatility hedged.

As Bizid, Jouini, and Koehl (1999) remark, the highly precise results associated with Perrakis and Ryan's pricing interval stem in part from the fact that the true transition probabilities render the price process of the equity claim near to that of a martingale under the true probability. Indeed, as the states of the world are assumed to be equiprobable, the ratio $E[p_T]/p_0$ is very close to one. In this example, the risk premium for the equity claim is approximately equal to 1% for one year, which is very low.

B. Non-Equiprobable States of the World

In the following example, we vary the value of the risk premium as in Bizid, Jouini, and Koehl (1999). We consider true probabilities that depend linearly on a parameter $\varepsilon \in \mathbb{R}$ as follows: $\pi_1 = ¼$, $\pi_2 = ¼ + \varepsilon$, $\pi_3 = ¼$, and $\pi_4 = ¼ - \varepsilon$. In this case, we can easily depict the risk premium as a function of the parameter ε (see Figure 2).

By definition, ε lies in the closed interval $[-0.0033, 0.25]$. Figure 3 presents the pricing bounds on a European call option (with $\sigma_{\min} = 10\%$, $\sigma_{\max} = 15\%$, and a 100-time steps tree) and varying ε. Perrakis' bounds widen continuously as ε increases (i.e., as the true probability moves away from the martingale case). For the equilibrium interval, however, this phenomenon is less important.

FIGURE 2

Risk Premium for the Stock vs. a Measure of the Risk Premium (parameter ε)

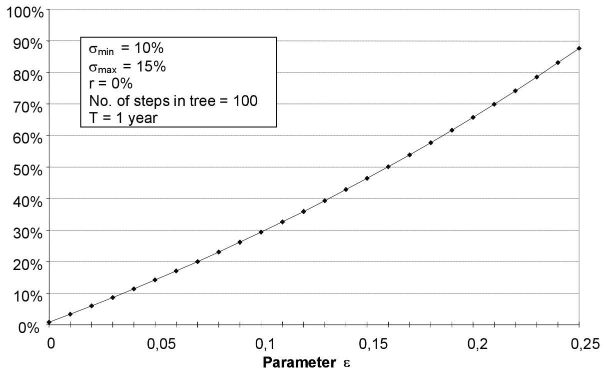

Figure 2 considers a quadrinomial stationary tree, representing the binomial volatility model. We assume that the probability, in the true world for the risky asset to realize a given state depends linearly on parameter ε. More specifically, we assume that, at each node, as ε increases to 1/4, the transition probability associated with the lowest possible value decreases to 0 and the transition probability of the second possible value increases to 1/2. As this parameter increases, the expected return (or, equivalently, the risk premium) of the underlying asset increases also (almost linearly). This means that, as ε increases, the true probability moves away from the martingale case (i.e., which has no risk premium). We represent here the risk premium when holding the risky asset for one year.

This example shows (at least heuristically) that i) in contrast to the complete markets case (Perrakis' (1986) interval), the precision of the equilibrium pricing interval seems to be less affected by the value of the equity claim's risk premium; ii) the completeness hypothesis does not permit one to improve the accuracy of the method for high values of the equity claim's risk premium; and iii) the equilibrium pricing interval is significantly more precise than the arbitrage pricing interval.

More precisely, Perrakis' (1986) pricing interval is, of course, always included in our equilibrium pricing interval because it supposes completeness of the market. The precision of Perrakis' pricing interval is linked to the discounted expected return of the equity claim at maturity. Owing to the risk aversion of the agents, this ratio is always greater than one.[3] Moreover, the nearer to one the ratio is, the finer is the precision of this interval. In an incomplete market, we find numerically that the nearer to one the ratio is, the wider is the relative precision of our equilibrium pricing interval with respect to Perrakis' pricing interval. However, empirical results suggest that with a high market risk premium, our bounds are equal to those of Perrakis. Therefore, our result can be interpreted as a justification of Perrakis' procedure for very risky market models.

VI. Conclusion

In this paper, we show that equilibrium conditions place strong restrictions on the admissible martingale measures and, further, on the asset prices compatible with a large class of utility functions: namely, the non-decreasing, concave, Von Neumann-Morgenstern utility functions. Our results are based upon ordering

[3]Aït-Sahalia and Lo (1998) propose this ratio as a measure of the market risk aversion.

FIGURE 3

Bounds of a European at-the-Money Call Option's Price vs. a Measure of the Risk Premium (parameter ε)

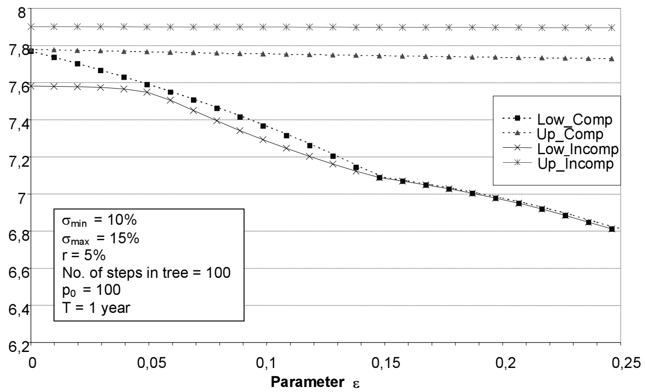

Figure 3 illustrates that the bounds of a European call option's price widen as ε (which defines the distribution of the true probability) increases. We consider more specifically an at-the-money call option with one year to maturity, in a stochastic volatility model, as a function of ε. In fact, only the lower bound of the call option's price varies much, because the call option price is an increasing function of the stock price. In this example, the constraint associated with the complete market case is $\varepsilon = 0.1483$. The constraint on the probabilities associated with the incomplete market case is $\varepsilon = 0.0485$. Note the exact match for the lower boundaries between an incomplete and a complete market if the parameter ε is greater than 0.1483.

properties on the state prices as in Perrakis (1986) and Perrakis and Ryan (1984) and involve the true probability as in Bizid, Jouini, and Koehl (1999).

We do not assume completeness of the financial market or the existence of any completion at the equilibrium (such an existence is equivalent to the equality of the marginal utilities over all agents). Our bounds are thus larger than Perrakis' (1986) bounds because the set of prices compatible with an equilibrium for a given asset in an incomplete market is larger than the set of all possible prices for that asset that are compatible with equilibrium in at least one completion.

Our bounds result from necessary conditions. The true equilibrium price interval (i.e., the set of prices that are compatible with an equilibrium in at least one specification of the utility function in the chosen class) could be slightly smaller than the set we obtain. However, we conjecture that these two sets are in fact equal.

Finally, throughout the paper, we assume that there is only one productive asset. However, it is easy to generalize our method to a model with many productive assets and with many derivatives on those assets. To do so, it suffices to replace the process p_t by the total wealth process or by the index process that we interpret as a proxy of the total wealth.

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