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General equilibrium with producers and brokers

Existence and regularity

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Abstract

In this paper we prove the existence of general equilibrium with transaction costs generalizing Hahn's (*Review of Economic Studies*, 1973, 40, 449-461) model by introducing producers and nonconvexities (in particular we allow for increasing returns in transaction sets). We also recover any exchange economy as a special case and this allows us to analyze the effects of small frictions on bid-ask prices, consumption vectors and utilities. We prove that, generically, the induced perturbations are of the same order as the frictions.

1. Introduction

Several authors have introduced transaction costs in a general equilibrium framework [in particular Hahn (1973) and Starrett (1973)]. In Hahn's model there are, besides consumers and producers, a special kind of producer: brokers. Brokers purchase at the ask price the goods supplied by consumers and producers, transform them according to a feasible transaction set and sell them back, at the bid price. In these papers, however, transaction sets are assumed to be convex cones. Hahn (1973) points out that this assumption is 'pretty terrible. [It] rules out increasing returns when causal observation suggests that set up costs are an important feature in transaction technologies'.

In this paper we are not making any convexity assumption on the production and transaction sets. This is, we believe, the typical case for transaction sets, as is pointed out in Hahn (1973) and

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Starrett (1973). Indeed, there are important fixed costs involved in transactions. For instance, in markets requiring immediacy such a fixed cost is time: the dealer must be present on the market at all times.

The main purpose of this paper is to study the effect of small frictions (in the transaction possibilities) on equilibrium. We show that, generically, at equilibrium the perturbations in prices, in the consumption vectors, and in the utility of the consumers is of the same order as the frictions introduced in the transaction possibilities.

In section 2 we introduce the general framework of this paper. In section 3 we establish an existence result of an equilibrium. In section 4 we study the effect of small frictions.

2. The model [1]

Let $\mathscr{E}$ be an economy with ℓ goods, $h = 1, \dots, \ell$, m consumers, $i = 1, \dots, m$, n producers, $j = 1, \dots, n$, and r brokers, $k = 1, \dots, r$.

The ith consumer is described by a continuous convex and locally non-satiated preferences preorder $\precsim_i$ on a consumption set $X_i = R^\ell_+$ and initial endowments ω^*_i. We denote by ω^* the vector of total initial endowments, i.e. $\omega^* = \Sigma^m_{i=1} \omega^*_i$. The jth producer is able to transform an input vector $y_j \in R^\ell_+$ into an output vector $y'_j \in R^\ell_+$, as long as (y_j, y'_j) belongs to the feasible production set $Y_j \subset R^{2\ell}_+$.

Transactions are carried out by the brokers. The kth broker buys $z_k \in R^\ell_+$ and sells $z'_k \in R^\ell_+$ constrained by $(z_k, z'_k) \in Z_k$, where Z_k is his feasible transaction set.

We assume that for all j (resp. all k) the production set Y_j (resp. transaction set Z_k) is non-empty, closed and $(Y_j + R^\ell_+ \times R^\ell_-) \cap R^{2\ell}_+ \subset Y_j$ [resp. $(Z_k + R^\ell_+ \times R^\ell_-) \cap R^{2\ell}_+ \subset Z_k$], i.e. less can be produced with more. We also assume that for every 'efficient' production (resp. transaction) plan $(y_j, y'_j) \in \partial Y_j$ [resp. $(z_k, z'_k) \in \partial Z_k$], the firm (resp. the broker) sets the price p in $\varphi_j(y_j, y'_j)$ [resp. $\psi_k(z_k, z'_k)$], where φ_j: $\partial Y_j \to S_{2\ell}$ (resp. ψ_k: $\partial Z_k \to S_{2\ell}$) is an upper semicontinuous [2] convex compact value correspondence called the pricing rule. For instance, this formalization takes into account profit maximization in the convex case and marginal pricing or cost-average pricing in general.

For a price system (p, q), production plans (y_j, y'_j) and transaction plans (z_k, z'_k) the revenue of the ith consumer is equal to $w_i(p, q, (y_j, y'_j), (z_k, z'_k))$, where (w_i) is a collection of positively homogeneous [with respect to (p, q)] mappings from $R^{2\ell}_+ \times R^{2\ell n} \times R^{2\ell r}$ into R satisfying $\Sigma^m_{i=1} w_i(p, q, (y_j, y'_j), (z_k, z'_k)) = \Sigma^n_{j=1} (-p \cdot y_j + q \cdot y'_j) + \Sigma^r_{k=1} (p \cdot z'_k - q \cdot z_k)$.

The ith consumer purchases a vector x_i at the price p and sells a vector x'_i at the price q constrained by $(x_i, x'_i) \in B(p, q, w_i)$, where $B(p, q, w_i) = \{\omega_i + x_i - x'_i \colon (x_i, x'_i) \in R^{2\ell}_+ \text{ and } p \cdot x_i - q \cdot x'_i \le w_i\}$. This merely says that his net consumption $\omega_i + x_i - x'_i$ must be in his budget set.

[1] If $x = (x_h)$ and $y = (y_h)$ are vectors in R^ℓ, we let $x \cdot y = \Sigma^\ell_{h=1} x_h y_h$ be the scalar product of R^ℓ, and $\|x\| = (x \cdot x)^{1/2}$ be the Euclidean norm. The notation $x \ge y$ (resp. $x \gg y$) means $x_h \ge y_h$ (resp. $x_h > y_h$) for all h; we let $R^\ell_+ = \{x \in R^\ell \colon x \ge 0\}$ and $R^\ell_{++} = \{x \in R^\ell \colon x \gg 0\}$. If $x \in R^\ell$ we denote by x^+ and x^- the vectors, respectively, with coordinates $x^+_h = \max(0, x_h)$ and $x^-_h = \max(0, -x_h)$. For $h = 1, \dots, \ell$ we denote by e^h the vector in R^ℓ with all coordinates equal to 0 except the hth coordinate which is equal to 1. We denote by e the vector in R^ℓ with all coordinates equal to 1 and by $S_\ell = \{p \in R^\ell_+ \colon \Sigma^\ell_{h=1} p_h = 1\}$. For $A \subset R^\ell$, we denote by cl$A$, ∂A and ri(A), respectively, the closure, the boundary and the relative interior of A.

[2] Given two topological spaces X and Y, a correspondence ϕ is said to be upper semi-continuous if ϕ is locally bounded and if the graph of ϕ, i.e. $\{(x, y) \in X \times Y \colon y \in \phi(x)\}$, is closed.

Definition 2.1. An equilibrium of $\mathscr{E} = ((X_i \preceq_i, w_i, \omega_i), (Y_j), (Z_k))$ is a list $(p, q, (x_i, x'_i), (y_j, y'_j), (z_k, z'_k))$ in $R_+^{2\ell} \times R_+^{2\ell m} \times R_+^{2\ell n} \times R_+^{2\ell r}$ such that:

(i) for all i, $\omega_i + x_i - x'_i$ is a best element for $\preceq_i$ in the budget set

$$B(p, q, w_i(p, q, (y_j, y'j), (z_k, z'k))) ;$$

(ii) for all j, $(y_j, y'_j) \in \partial Y_j$ and $(p, q) \in \varphi_j(y_j, y'_j)$;

(iii) for all k, $(z_k, z'_k) \in \partial Z_k$ and $(p, q) \in \psi_k(z_k, z'_k)$;

(iv) $\begin{cases} \Sigma_{i=1}^m x_i + \Sigma_{j=1}^n y_j \leq \Sigma_{k=1}^r z'_k , \\ \Sigma_{i=1}^m x'_i + \Sigma_{j=1}^n y'_j \geq \Sigma_{k=1}^r z_k ; \end{cases}$

(v) $\begin{cases} p \cdot (\Sigma_{i=1}^m x_i + \Sigma_{j=1}^n y_j - \Sigma_{k=1}^r z'_k) = 0 , \\ q \cdot (\Sigma_{i=1}^m x'_i + \Sigma_{j=1}^n y'_j - \Sigma_{k=1}^r z_k) = 0 . \end{cases}$

For $\omega \in R\ell$ let $A(\omega) = \{((y_j, y'_j), (z_k, z'_k)) \in \Pi_{j=1}^n \ \partial Y_j \times \Pi_{k=1}^r \ \partial Z_k\colon \ \Sigma_{j=1}^n y'_j - \Sigma_{j=1}^n y_j + \Sigma_{k=1}^r z'_k - \Sigma_{k=1}^r z_k + \omega \geq 0\}$ be the set of attainable production and transaction plans. We can introduce at this stage the following boundedness and survival assumptions.

Assumption (B). For every $\omega \geq \omega^*$, the attainable set $A(\omega)$ is bounded.

Assumption (WSA). For every $\omega \geq \omega^*$, the following assumption $(SA(\omega))$ holds:
For every $(p, q, (y_j, y'_j), (z_k, z'_k)) \in R_+^{2\ell} \times A(\omega)$ such that $p \geq q$ and such that $(p, q) \in (\bigcap_{j=1}^k \varphi_j(y_j, y'_j)) \cap (\bigcap_{k=1}^r \psi_k(z_k, z'_k))$ we have for all i, $w_i(p, q, (y_j, y'_j), (z_k, z'_k)) + q \cdot \omega_i > 0$.

Assumption (WSA) means that for an economy with initial endowments ω at least equal to ω^*, the ith consumer can generate a positive total revenue by selling his endowment (at the price q) for all production–transaction marginal pricing equilibria that are in the attainable set.

3. The existence result

In this section we state and prove the existence of an equilibrium as defined in section 2. In fact, using the results of Bonnisseau and Corne (1991) we prove the existence of a marginal pricing equilibrium. Also, as we shall underline in the proof, we can show the existence of an equilibrium for general pricing rules with bounded losses using Bonnisseau and Cornet (1988).

We recall that, for a closed set $Y \in R^\ell$, a perpendicular vector to Y at $y \in Y$ is an element in the set

$$\perp_Y(y) = \{p \in R^\ell\colon \exists \rho > 0, \ \forall y' \in Y, \ p \cdot y \geq p \cdot y' - \rho \|y - y'\|^2\} .$$

Then, Clarke's normal cone to Y at y [see Clarke (1983)], denoted $N_Y(y)$, is the closed convex hull of the set

$$\{p \in R^\ell\colon \exists (y_q) \subset Y, (y_q) \to y, \ \exists (p_q) \subset R^\ell, \ (p_q) \to p \text{ and } \forall q, \ p_q \in \perp_Y(y)\} .$$

As in Guesnerie (1975) and Bonnisseau and Cornet (1991) the marginal pricing rule consists in fulfilling the first-order necessary condition of profit maximization. Formally, this can be stated: $(p, q) \in \varphi_j(y_j, y'_j)$ if $(-p, q) \in N_{Y_j}(y_j, y'_j)$ and $(p, q) \in \psi_k(z_k, z'_k)$ if $(-q, p) \in N_{Z_k}(z_k, z'_k)$. This pricing rule is of interest because the second welfare theorem holds for a marginal pricing production economy. In our case this result continues to hold, in spite of transaction costs, because the definition of Pareto optimality ought to take the transaction possibilities into account as well as the production possibilities.

Theorem 3.1. Under Assumptions (B) and (WSA), the transaction cost economy $\mathscr{E} = ((X_i, \leq_i, w_i, \omega_i), (Y_j), (Z_k))$ *admits a marginal pricing equilibrium and we have* $p \geq q$.

Proof. For $j = 1, \ldots, n$, $k = 1, \ldots, r$ and $i = 1, \ldots, m$, let

$$\tilde{Y}_j = \{(-y_j, y_j') \in R^{2\ell} : (y_j, y_j') \in Y_j\} - \{0\} \times R_+^\ell ,$$

$$\tilde{Z}_k = \{(z_k', -z_k) \in R^{2\ell} : (z_k, z_k') \in Z_k\} - R_+^\ell \times \{0\} ,$$

$$\tilde{X}_i = \{(x_i, -x_i') \in R^{2\ell} : x_i \geq 0,\ x_i' \geq 0 \text{ and } x_i - x_i' + \omega_i^* \geq 0\} ,$$

and let $\tilde{\leq}_i$ be the natural preorder induced by $\leq_i$ [i.e. $(x_i, -x_i' + \omega_i^*) \tilde{\leq}_i (\bar{x}_i, -\bar{x}_i' + \omega_i^*)$ if $(x_i - x_i' + \omega_i^*) \leq_i (\bar{x}_i - \bar{x}_i' + \omega_i^*)$].

Let $\tilde{\mathscr{E}}$ be the economy, with 2ℓ goods, defined by $((\tilde{X}_i, \tilde{\leq}_i, \tilde{w}_i, \tilde{w}_i^*), (\tilde{Y}_j), (\tilde{Z}_k))$, where for $i = 1, \ldots, m$, $\tilde{\omega}_i^* = (0, 0)$, where $((\tilde{Y}_j), (\tilde{Z}_k))$ are the production sets of this economy and where $\tilde{w}_i$ is defined by the formula: $\tilde{w}_i(p, q, (-y_j, y_j'), (z_k', -z_k)) = w_i(p, q, (y_j, y_j'), (z_k, z_k'))$.

In this transformed economy, producers and brokers are indistinguishable in their behavior: they take (p, q) for the price vector.

All the assumptions of Bonnisseau and Cornet (1991) are satisfied and guarantee the existence of a marginal pricing equilibrium $(\tilde{p}, \tilde{q}, (\tilde{x}_i, -\tilde{x}_i'), (-\tilde{y}_j, \tilde{y}_j'), (\tilde{z}_k', -\tilde{z}_k))$ for $\tilde{\mathscr{E}}$. We can then easily verify that $(\tilde{p}, \tilde{q}, (\tilde{x}_i, \tilde{x}_i'), (\tilde{y}_j, (\tilde{y}_j')^+), (\tilde{z}_k, (\tilde{z}_k')^+))$ is an $\mathscr{E}$-equilibrium. □

4. Small frictions in an exchange economy

In this section we analyze the effect of small frictions on an exchange economy. The definitions of the previous section extend straightforwardly to this case. The results that we obtain could easily be generalized to production economies.

Let $\mathscr{E} = (X_i, \leq_i, \omega_i)$ be an exchange economy with $(\omega_i) \gg 0$. We shall see that its equilibria can be recovered as equilibria of a transaction economy $\hat{\mathscr{E}}$ obtained from $\mathscr{E}$ by adjunction of a transaction set $Z_0 = \{(z, z') \in R_+^{2\ell} : z' \leq z \text{ and } z' \leq 2\omega\}$. Note that since Z_0 is convex the first-order condition for profit maximization is also sufficient and the profits of the broker are always non-negative. We assume that these profits are evenly distributed across consumers.[3] The economy $\hat{\mathscr{E}}$ satisfies all the assumptions of our existence result.

Proposition 4.1. Let $(p, (x_i))$ *be an equilibrium of the exchange economy* $\mathscr{E} = (X_i, \leq_i, \omega_i)$, *then* $(p, p, ((x_i - \omega_i)^+, (x_i - \omega_i)^-), (\Sigma_{i=1}^m (x_i - \omega_i)^-, \Sigma_{i=1}^m (x_i - \omega_i)^+))$ *is an equilibrium of the transaction economy* $\hat{\mathscr{E}} = ((X_i, \leq_i, \omega_i), Z_0)$. *Conversely, if* $(p, q, (x_i, x_i'), (z, z'))$ *is an* $\hat{\mathscr{E}}$*-equilibrium, then* $p = q$ *and* $(p, (x_i - x_i' + \omega_i))$ *is an* $\mathscr{E}$*-equilibrium.*

The proof of this proposition is left to the reader.

From now on we shall identify an exchange economy with its associated transaction economy

[3] This assumption is in fact innocuous since at the $\mathscr{E}$-equilibria the broker always makes zero profits.

obtained by adjunction of the transaction set Z_0. Let us consider now a transaction economy without producers and with a transaction set Z_α sufficiently 'close' to Z_0.

Let Z_n be a sequence of convex subsets of $R^{2\ell}$ and let π_n be the projection mapping on Z_n. The sequence (Z_n) is said to converge to a convex subset Z of $R^{2\ell}$ if the sequence π_n converges to the projection mapping π on Z for the $\mathscr{C}^2$ norm on every compact subset of $R^{2\ell}\backslash A$, where A is a null measure subset of $R^{2\ell}$. This topology on transaction–production sets is the same as in Mas-Colell (1985, section 3.8).

For our purpose we shall strengthen the assumption on the preferences assuming that, as in Balasko (1988, section 2.3), the preordering $\preceq_i$ is representable by a quasi-concave smooth utility function u_i for which the first-order sufficient condition for strict monotonicity and the second-order sufficient condition for strict quasi-concavity are satisfied. What follows can, however, be generalized to non-convex transaction sets noting that, under the free disposal assumption, $\partial\tilde{Z}_n$ appears as the graph of a function λ_n: $e^\perp \rightarrow R$ and hence π_n can be replaced by λ_n in the definition of convergence.

In the following theorem, we only consider convex perturbations of the set Z_0 in order to apply Mas-Colell's (1985) regularity results. There is no real difficulty in extending our result to non-convex perturbations using regularity results as in Jouini (1992a,b).

Theorem 4.2. For almost every initial endowment $\omega \in R_+^{m\ell}$, and for every exchange equilibrium of the economy $\hat{\mathscr{E}} = ((X_i, \preceq_i, \omega_i), Z_0)$ if we perturb slightly Z_0 we can find near this equilibrium an equilibrium of the perturbed economy.

This means, in particular, that by introducing small frictions into an exchange economy we obtain consumption vectors and bid–ask prices in a neighborhood of the exchange equilibrium. Consequently the bid–ask spread is small. Since our argument relies on an implicit function theorem, frictions induce exactly first-order perturbations in prices and allocations. Furthermore, the difference in utility for the ith consumer is of second or higher order only if consumers have colinear net trade vectors, which is easily shown to be non-generic.

Note that, in general, this result does not hold for *all* initial endowments ω since if $\mathscr{E}$ is a singular exchange economy [see Dierker (1982) and Mas-Colell (1985)] it is then easy to see that slight perturbations of the parameters of the economy can induce jumps in the equilibrium prices. Hence, using Balasko's lemma (1988, lemma 5.2.1) we can show that these jumps in prices induce jumps in utilities and consequently in allocations.

If the economy $\mathscr{E} = (X_i, \preceq_i, \omega_i)$ is a regular exchange economy [see Dierker (1982), Mas-Colell (1985) and Balasko (1988)], then it admits a finite number of equilibria. If for all equilibria $(p, (x_i))$ of $\mathscr{E}$ and $h = 1, \ldots, \ell$ we have $x_i^h \neq \omega_i^h$, we say that the economy $\mathscr{E}$ is transactive.

Lemma 1. If $m \geq 2$, for almost every $\omega \in R_+^{m\ell}$ the economy $\mathscr{E} = (X_i, \preceq_i, \omega_i)$ is a regular transactive economy.

The proof of this lemma is obtained by classical arguments of differential topology. Let us now define the demand correspondence $\hat{D}_i$ of the ith consumer when facing the price vector (p, q), with $p \gg q$, as the set of solutions (x_i, x_i') of the following program:

$$(\mathcal{P})\begin{cases}\max u_i(\omega_i + x_i - x'_i)\ ,\\ x_i \geq 0\ ,\\ x'_i \geq 0\ ,\\ p\cdot x_i - q\cdot x'_i \leq 0\ .\end{cases}$$

It is easy to show that in this case the solutions (x_i, x'_i) must satisfy $x_i \cdot x'_i = 0$.

We extend the previous definition to the case where $p \geq q$ by adding the condition $x_i \cdot x'_i = 0$. The only effect of this constrain is to get rid of the trivial indeterminacy of solutions when $p_h = q_h$ for some h. In fact, strict quasi-concavity of u_i implies that $\hat{D}_i$ is a well-defined function for $p \geq q$.

Lemma 2. Let $(p^*, p^*) \in \mathrm{ri}(S_{2\ell})$ *such that* $\hat{D}_i(p^*, p^*) = (\hat{x}_i, \hat{x}'_i)$ *with* $\hat{x}_i + \hat{x}'_i \gg 0$. *Then* $\hat{D}_i$ *is* $\mathscr{C}^1$ *in a neighborhood of* (p^*, p^*).

This lemma is a consequence of a classical implicit function theorem.

Proof of Theorem 4.2. The result is obvious for $m = 1$. Assume now that $m \geq 2$ and let $\omega \in R^{m\ell}_{++}$ such that $\mathscr{E} = (X_i, \preceq_i, \omega_i)$ is a regular transactive economy (by Lemma 1 this is true for almost every ω). In fact, in what follows we shall assume that the consumers are directly characterized by their demand functions and we shall define $\mathscr{E}$ by (X_i, D_i, ω_i). Let $\hat{\mathscr{E}} = ((X_i, \hat{D}_i, \omega_i), Z_0)$ and $\tilde{\mathscr{E}}$ be the economy obtained from $\hat{\mathscr{E}}$.

Since $\hat{Z}_0$ is a polyhedral cone of the activity matrix $A = {}^t(-I_\ell, I_\ell)$ and the ith demand function $\tilde{D}_i$ in the transformed economy $\tilde{\mathscr{E}}$ is defined by $\tilde{D}_i(p, q) = (x_i, -x'_i)$, where $(x_i, x'_i) = \hat{D}_i(p, q)$, we have, following Mas-Colell (1985, Proposition 6.4.1) that $\tilde{\mathscr{E}}$ is regular if for all equilibrium of $\tilde{\mathscr{E}}$ the matrix

$$M(p, q) = \begin{pmatrix} -\partial\tilde{D}_i(p, q) & A & {}^t(p, q) \\ -{}^tA & 0 & 0 \\ -(p, q) & 0 & 0 \end{pmatrix}$$

has full rank, which is easily verified. We then obtain the result of Theorem 4.2 as a direct consequence of Mas-Colell's results (1985, section 6.4). □

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